Linear Relations in the Calkin Algebra for Composition Operators

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Abstract

We consider this and related questions: When is a finite linear combination of composition operators, acting on the Hardy space or the standard weighted Bergman spaces on the unit disk, a compact operator?

1 Introduction

For $\beta \geq 1$, let $\mathcal{D}_\beta$ denote the reproducing kernel Hilbert space of functions analytic in the unit disk $D = \{z : |z| < 1\}$ and having the kernel functions $k_w(z) = (1 - \overline{w}z)^{-\beta}$. Thus, $f(w) = \langle f, k_w \rangle$ for w in D and f in $\mathcal{D}_\beta$. The Hardy space H^2 is exactly $\mathcal{D}_1$ and when $\beta > 1$, $\mathcal{D}_\beta$ is the standard weighted Bergman space A^2_α with $\alpha + 2 = \beta$, see Section 2.1. We consider composition operators $C_\varphi : f \to f \circ \varphi$ acting on $\mathcal{D}_\beta$, where φ is an analytic self-map of D. When $\beta \geq 1$, every C_φ lies in $\mathcal{B}(\mathcal{D}_\beta)$, the algebra of bounded operators on $\mathcal{D}_\beta$. Unlike the classes of Toeplitz and Hankel operators which act on $\mathcal{D}_\beta$, the set of composition operators in $\mathcal{B}(\mathcal{D}_\beta)$ has no obvious additive or linear structure. However, in the Bergman space case $\beta > 1$, the second author observed additive structure modulo the ideal $\mathcal{K}$ of compact operators and characterized those pairs φ and ψ for which $C_\varphi - C_\psi$ is compact [16]. Our purpose here is twofold: to present some analogous results for the H^2 case $\beta = 1$, and to pass from additive to linear structure in the Calkin algebra $\mathcal{B}(\mathcal{D}_\beta)/\mathcal{K}$.

Composition operators which are themselves compact were characterized in the A^2_α case by MacCluer and Shapiro [15] and on H^2 by Shapiro [25]; in [22] Sarason found a different condition, sufficient for H^2 and necessary and sufficient for L^1, later shown by Shapiro and Sundberg [27] to be necessary in the H^2 case as well. The problem

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of compact difference was raised in explicit form by Shapiro and Sundberg [26] and MacCluer [13]; these authors found several criteria, some necessary and some sufficient. More recently, MacCluer, Ohno, and Zhao [14] have shown that compactness of $C_\varphi - C_\psi$ acting on H^∞, the space of bounded analytic functions on D, is characterized in terms of the quantity $\rho(z) = \left|\frac{\varphi(z)-\psi(z)}{1-\overline{\varphi(z)}\psi(z)}\right|$, the pseudo-hyperbolic distance between values of φ and ψ. The second author of the present article subsequently found that ρ plays a related role on the Bergman spaces, as follows:

Theorem A [16] *Suppose that φ and ψ are analytic self-maps of D and $\alpha > -1$. Then $C_\varphi - C_\psi$ is a compact operator on A^2_α if and only if*

$$\lim_{|z|\to 1} \rho(z) \left\{ \frac{1-|z|^2}{1-|\varphi(z)|^2} + \frac{1-|z|^2}{1-|\psi(z)|^2} \right\} = 0.$$

Let $F(\varphi)$ denote the set of points in the unit circle ∂D at which φ has a finite angular derivative in the sense of Caratheodory, see Section 2.2. Also, we will use the notation $A \equiv B \pmod{\mathcal{K}}$ to indicate that two bounded operators A and B have compact difference. There is a "sum theorem," as follows:

Theorem B [16] *Let $\varphi, \varphi_1, \dots, \varphi_n$ be analytic self-maps of D for which the sets $F(\varphi_i)$, $i = 1, \dots, n$, are pairwise disjoint and with $F(\varphi) = F(\varphi_1) \cup \cdots \cup F(\varphi_n)$. Consider $C_\varphi, C_{\varphi_1}, \dots, C_{\varphi_n}$ as acting on A^2_α where $\alpha > -1$, and let*

$$\rho_i = \left| \frac{\varphi - \varphi_i}{1 - \overline{\varphi}\varphi_i} \right|, \quad i = 1, \dots, n.$$

Then the following are equivalent:

(i) For each $i = 1, \dots, n$ and each ζ in $F(\varphi_i)$,

$$\lim_{z\to\zeta} \rho_i(z) \left\{ \frac{1-|z|^2}{1-|\varphi(z)|^2} + \frac{1-|z|^2}{1-|\varphi_i(z)|^2} \right\} = 0.$$

(ii) $C_\varphi \equiv C_{\varphi_1} + \cdots + C_{\varphi_n} \pmod{\mathcal{K}}$.

The following useful modification of Theorem A, which localizes the notion of compact difference, is implicit in [16].

Theorem C *Suppose that φ and ψ are analytic self-maps of D, $\alpha > -1$ and G is a measurable subset of D. If*

$$\lim_{|z|\to 1} \chi_G(z)\rho(z) \left\{ \frac{1-|z|^2}{1-|\varphi(z)|^2} + \frac{1-|z|^2}{1-|\psi(z)|^2} \right\} = 0,$$

then $M_{\chi_G}(C_\varphi - C_\psi)$ is a compact operator from A^2_α into its containing L^2 space. Here M_{χ_G} denotes the operator of multiplication by the characteristic function χ_G.

Other recent related work includes the following: Bourdon, Levi, Narayan, and J. H. Shapiro [4] show that a composition operator associated with an "almost linear fractional" map is, in fact, a compact perturbation of a linear fractional composition operator; Bourdon [3] treats the question of compact difference vs. topological connectedness for linear fractional maps (see Section 6 below); J. E. Shapiro [24] shows that if $C_\varphi - C_\psi$ is compact on H^2, the singular parts of the Clark measures of φ and ψ coincide (see Sect. 2.3). Most recently, Nieminen and Saksman [19] have shown that the just-quoted condition of J. E. Shapiro on singular parts, plus uniform integrability of the differences of the absolutely continuous densities of the respective Clark measures of φ and ψ, are together equivalent to compactness of $C_\varphi - C_\psi$ on L^1 (or, on the space of complex Borel measures on $\partial\mathbb{D}$) and are thus sufficient on H^2. In a different direction, Gorkin and Mortini [10] have characterized compactness for finite linear combinations of composition operators acting on uniform algebras.

Our analogues of Theorems A, B, and C for H^2 require some different methods. For us the key is an application of Clark measures, following ideas of Sarason [22], Cima and Matheson [5] and J. E. Shapiro [24]. In Section 3 we obtain essential norm estimates for weighted composition operators on H^2 analogous to the Cima-Matheson essential norm formula for (unweighted) composition operators [5]. We combine these results in Section 4 with a general principle of Moorhouse and Toews [17] and Carleson measure estimates as in [16] to obtain our H^2 results.

Section 5 is devoted to the question of when a given finite linear combination of composition operators is compact. We look at lower bounds, given in terms of first- and higher-order boundary data, for the essential norm of a linear combination; these results further develop ideas of MacCluer [13]. We introduce the class $\mathcal{S}$ of analytic self-maps φ of D having "sufficient data" at every point in ∂D where φ makes significant contact with the boundary. For φ and ψ in $\mathcal{S}$ we make clear the obstructions to the essential norm $\|C_\varphi - C_\psi\|_e$ being small, the conditions under which it must be small, and when it is zero, that is, when $C_\varphi - C_\psi$ is compact. For $\varphi_1, \ldots, \varphi_n$ in $\mathcal{S}$, we characterize, via a finite system of linear equations involving boundary data of these maps, those coefficients for which $c_1 C_{\varphi_1} + \cdots + c_n C_{\varphi_n}$ is compact. An application is a simple algorithm for determining the dimension of the vector space in $\mathcal{B}(\mathcal{D}_\beta)/\mathcal{K}$ spanned by the cosets $[C_{\varphi_1}], \ldots, [C_{\varphi_n}]$.

The final Section 6 concerns a problem first studied by Berkson [2], subsequently considered by MacCluer [13], Shapiro and Sundberg [26], and most recently by Toews and the second author [17] and Bourdon [3]: to characterize those pairs φ and ψ for which C_φ and C_ψ lie in the same connected component of the topological space of composition operators, equipped with the norm topology on $\mathcal{B}(\mathcal{D}_\beta)$. We observe that a general sufficient condition of the second author for the Bergman space case [16] extends to H^2 and apply this result to those C_φ with φ lying in a certain subclass $\mathcal{S}_0$ of $\mathcal{S}$ to determine when C_φ and C_ψ lie in the same component of $\{C_\varphi : \varphi \in \mathcal{S}_0\}$.

A variation on ideas of Berkson [2], Shapiro and Sundberg [26], and MacCluer [13] (see Exercise 9.3.2 in [8]) states that if $\varphi_1, \ldots, \varphi_n$ are analytic self-maps of D, and if

$J(\varphi)$ denotes the set of points $e^{i\theta}$ in ∂D with $|\varphi(e^{i\theta})| = 1$, then

$$\left\| \sum_{j=1}^{n} c_j C_{\varphi_j} \right\|_e^2 \geq \frac{1}{2\pi} \sum_{j=1}^{n} |c_j|^2 |J(\varphi_i)|,$$

where $|J(\varphi)|$ is the arclength measure of $J(\varphi)$. Accordingly, to study the questions discussed above, we assume throughout that our analytic self-maps φ of D satisfy $|\varphi(e^{i\theta})| < 1$ a.e.

2 Preliminaries

Here we collect some preliminary facts used in the sequel.

2.1 The Hardy and Bergman Spaces

The Hardy space $H^2 = \mathcal{D}_1$ is the set of all functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ analytic in D with

$$\|f\|^2 \equiv \sum_{n=0}^{\infty} |a_n|^2 < \infty.$$

Given f in H^2, the non-tangential limit $f(e^{i\theta}) = \lim_{\angle z \to e^{i\theta}} f(z)$ exists for $d\theta$-almost every $e^{i\theta}$ in ∂D. Moreover, the correspondence $f(z) \to f(e^{i\theta})$ allows one to think of H^2 as the closed subspace of $L^2 = L^2\left(\partial D, \frac{d\theta}{2\pi}\right)$ with orthonormal basis $\{e^{in\theta}\}_{n=0}^{\infty}$.

For $\alpha > -1$, the Bergman space A_α^2 is the set of functions f analytic in D with

$$\|f\|^2 = \frac{\alpha+1}{\pi} \int_D |f(z)|^2 (1-|z|^2)^\alpha dA(z) < \infty,$$

where dA is Lebesgue area measure on D. As mentioned above, $\mathcal{D}_\beta = A_{\beta-2}^2$ for $\beta > 1$, with equality of norms.

For information about H^2 and A_α^2, see [9] and [8].

2.2 Angular Derivatives

Let φ be an analytic self-map of D. Then φ has a (finite) angular derivative at ζ in ∂D provided $\varphi(\zeta)$, the non-tangential limit of φ at ζ, exists and has modulus one, and

$$\varphi'(\zeta) \equiv \lim_{\angle z \to \zeta} \frac{\varphi(z) - \varphi(\zeta)}{z - \zeta}$$

exists as a finite complex number. If the angular derivative $\varphi'(\zeta)$ fails to exist, we write $|\varphi'(\zeta)| = \infty$. In either case the Julia-Caratheodory Theorem [8] asserts in part that

$$\liminf_{z \to \zeta} \frac{1 - |\varphi(z)|}{1 - |z|} = |\varphi'(\zeta)|,$$

where the limit inferior is taken unrestrictedly in D; moreover $|\varphi'(\zeta)| > 0$. Throughout we write $F(\varphi)$ for the set of all points in ∂D where φ has a finite angular derivative. For ζ in $F(\varphi)$ we have the relation $\varphi'(\zeta) = \overline{\zeta}\varphi(\zeta)|\varphi'(\zeta)|$. A condition necessary for the composition operator C_φ to act compactly on H^2 is that $F(\varphi)$ be empty [28]. On the Bergman space A^2_α, this condition is both necessary and sufficient [15].

2.3 Clark Measures

Let φ be an analytic self-map of D. If $|\alpha| = 1$, there exists a finite positive Borel measure μ_α on ∂D such that

$$\frac{1-|\varphi(z)|^2}{|\alpha-\varphi(z)|^2} = \mathrm{Re}\left(\frac{\alpha+\varphi(z)}{\alpha-\varphi(z)}\right) = \int_{\partial D} P_z(e^{it})d\mu_\alpha(t) \tag{2.1}$$

for z in D, where

$$P_z(e^{it}) = \frac{1-|z|^2}{|e^{it}-z|^2}$$

is the Poisson kernel at z. The existence of μ_α follows since the left side of equation (2.1) is a positive harmonic function. The measures μ_α (the *Clark measures* of φ) were introduced as an operator-theoretic tool by D. N. Clark [6], and have been further analyzed by Alexsandrov [1], Poltoratski [20], and Sarason [23].

On decomposing $\mu_\alpha = \mu_\alpha^{ac} + \mu_\alpha^s$, where μ_α^{ac} and μ_α^s are, respectively, the absolutely continuous and singular parts with respect to Lebesgue measure, one finds by Fatou's theorem [9] that

$$\mu_\alpha^{ac} = \frac{1-|\varphi(e^{i\theta})|^2}{|\alpha-\varphi(e^{i\theta})|^2}\,\frac{d\theta}{2\pi}.$$

The singular part μ_α^s is carried by $\varphi^{-1}(\{\alpha\})$, the set of those ζ in ∂D at which $\varphi(\zeta)$ exists and equals α, and is itself the sum of the pure point measure

$$\mu_\alpha^{pp} = \sum_{\varphi(\zeta)=\alpha} \frac{1}{|\varphi'(\zeta)|}\,\delta_\zeta \tag{2.2}$$

(here δ_ζ is the unit point mass at ζ) and a continuous singular measure μ_α^{cs}, either of which can vanish.

Let us write

$$E(\varphi) = \overline{\bigcup_{|\alpha|=1} spt(\mu_\alpha^s)},$$

where $spt(\mu)$ denotes the closed support of a measure μ. It is clear from Eqn. (2.2) that $F(\varphi)$ is a subset of $E(\varphi)$.

2.4 Essential Norms

Let H and L be separable Hilbert spaces and write $\mathcal{B}(H, L)$ for the space of bounded operators from H to L. Let $\mathcal{K}$ denote the subspace of compact operators in $\mathcal{B}(H, L)$. The essential norm $\|T\|_e$ of an operator T in $\mathcal{B}(H, L)$ is the operator-norm distance from T to $\mathcal{K}$. We will find this alternate description useful:

$$\|T\|_e = \sup_{\{f_n\}\in\mathcal{U}} \left(\limsup_{n\to\infty} \|Tf_n\| \right), \tag{2.3}$$

where $\mathcal{U}$ is the collection of all sequences $\{f_n\}$ of unit vectors in H which tend to zero weakly.

2.5 Carleson Measures

For a point ζ on the unit circle and $\delta > 0$, let $S(\zeta, \delta) = \{z \in D : \ |\zeta - z| < \delta\}$. If μ is a finite positive Borel measure on D and $\beta \geq 1$, we consider the quantities

$$\Delta_\beta(\mu) = \sup_{|\zeta|=1, \delta>0} \frac{\mu(S(\zeta,\delta))}{\delta^\beta}, \qquad \Delta^*_\beta(\mu) = \limsup_{\delta\to 0} \left\{ \sup_{|\zeta|=1} \frac{\mu(S(\zeta,\delta))}{\delta^\beta} \right\}. \tag{2.4}$$

One often says μ is a β-*Carleson* measure if $\Delta_\beta(\mu) < \infty$, and a *vanishing* β-*Carleson* measure if $\Delta^*_\beta(\mu) = 0$. The reader might consult [8] for the history of the following well-known result. The statement about $\|J\|_e$ can be deduced from ideas in the proof of Theorem 3.12 in [8].

Theorem 2.1 *Let μ be a finite positive Borel measure on D and assume $\beta \geq 1$. Then:*

(i) The space $\mathcal{D}_\beta$ (considered as a space of analytic functions on D) is contained in $L^2(\mu)$ if and only if μ is a β-Carleson measure. In this case the inclusion map $J : \mathcal{D}_\beta \to L^2(\mu)$ is bounded with norm comparable to $\sqrt{\Delta_\beta(\mu)}$.

(ii) If $\mathcal{D}_\beta$ is contained in $L^2(\mu)$, then $\|J\|_e$, the essential norm of the inclusion map, is comparable to $\sqrt{\Delta^*_\beta(\mu)}$. In particular, J is compact if and only if μ is a vanishing β-Carleson measure.

2.6 A General Scheme for Compact Difference and Arc-Connectedness

For a bounded analytic function w on ∂D, one can form the associated multiplication operator $M_w : f \to wf$. If φ is an analytic self-map of D, then we have the weighted composition operator $M_w C_\varphi$. Given two analytic self-maps of D, φ, and ψ, consider the self-maps $\varphi_t = t\varphi + (1-t)\psi$, $0 \leq t \leq 1$. Based on the formal operator identity

$$C_{\varphi_s} - C_{\varphi_r} = M_{\varphi-\psi} \left[\int_r^s C_{\varphi_t} dt \right] X, \tag{2.5}$$

$0 \le r < s \le 1$, and the fact that the differentiation operator $X = \frac{d}{dz}$ is a topological isomorphism of H_0^2 (the subspace of H^2 consisting of all functions that vanish at the origin) and the Bergman space A_1^2 (see [8]), the second author and C. Toews proved the following:

Theorem 2.2 [17] *Let φ, ψ, and φ_t, $0 \le t \le 1$, be as above.*

(i) Suppose the weighted composition operators $M_{\varphi-\psi}C_{\varphi_t}$ act boundedly from A_1^2 to H^2, with uniformly bounded norms, $0 \le t \le 1$. Then, there is a constant $B > 0$ such that, as operators on H^2,

$$\|C_{\varphi_s} - C_{\varphi_r}\| \le B|s-r|, \ \ 0 \le r < s < 1.$$

(ii) Suppose that φ, ψ, and φ_t satisfy the hypotheses of part (i) above, and in addition, that for each t, $0 \le t \le 1$, $M_{\varphi-\psi}C_{\varphi_t}$ is a compact operator from A_1^2 to H^2. Then $C_\varphi - C_\psi$ is a compact operator on H^2.

The above result remains true if one replaces H^2 and A_1^2 by A_α^2 and $A_{\alpha+2}^2$, respectively, where $\alpha > -1$, see [17].

3 Weighted composition operators on H^2 and L^2

For the analytic self-maps of D considered here (those with $|\varphi(e^{i\theta})| < 1$ a.e. on ∂D), Sarason [22] found a convenient representation of C_φ as an integral operator on H^2 and even on the larger space $L^2 = L^2\left(\partial D, \frac{d\theta}{2\pi}\right)$. For f in L^2, extend f to a harmonic function in D via the Poisson integral: $f(z) = \int_{\partial D} P_z(e^{it}) f(e^{it}) \frac{dt}{2\pi}$. Putting $(C_\varphi f)(e^{i\theta}) = f(\varphi(e^{i\theta}))$ using the extended f (since $|\varphi(e^{i\theta})| < 1$ a.e.), one has

$$(C_\varphi f)(e^{i\theta}) = \int_{\partial D} \frac{1 - |\varphi(e^{i\theta})|^2}{|e^{it} - \varphi(e^{i\theta})|^2} \, f(e^{it}) \frac{dt}{2\pi}. \tag{3.1}$$

Using the Schur test for boundedness of integral operators (stated below), Sarason showed that C_φ is compact on H^2 if the Clark measures μ_α of φ are absolutely continuous for all α in ∂D; J. H. Shapiro and C. Sundberg [27] established the converse via function-theoretic methods. Subsequently Cima and Matheson [5] discovered the following expression for the essential norm of an arbitrary C_φ acting on H^2:

$$\|C_\varphi\|_e^2 = \sup_{|\alpha|=1} \mu_\alpha^s(\partial D), \tag{3.2}$$

a formula foreshadowed by C. Cowen's inequalities for smooth φ [7, p. 84].

Here we adapt the integral operator approach to investigate essential norms of *weighted* composition operators

$$M_w C_\varphi : \ f \to w \cdot (f \circ \varphi).$$

We allow w in L^∞ and consider $M_w C_\varphi$ as mapping L^2 to L^2, H^2 to L^2 or (in the event that w is in H^∞), H^2 to H^2. In all cases we have the following:

Theorem 3.1 *Let φ be an analytic self-map of D with $|\varphi(e^{i\theta})| < 1$ a.e. and having Clark measures μ_α, $|\alpha| = 1$. Suppose w is a bounded measurable function on ∂D such that $|w|$ is continuous at every point of $E(\varphi)$. Then*

$$\sup_{|\alpha|=1} \int_{\partial D} |w|^2 d\mu_\alpha^s \leq \|M_w C_\varphi\|_e^2 \leq 4 \sup_{|\alpha|=1} \int_{\partial D} |w|^2 d\mu_\alpha^s.$$

In particular, $M_w C_\varphi$ is compact if and only if $w \equiv 0$ on $E(\varphi)$.

For the proof, our essential tool is the following.

The Schur Test [18, p. 282] *Consider two measure spaces (X, μ) and (Y, ν), and let N be a measurable function on the product space $Y \times X$. Suppose there exist positive measurable functions p on X and q on Y and constants $A, B > 0$ satisfying*

$$\int_X |N(y,x)| p(x) d\mu(x) \leq A q(y), \ y \text{ in } Y, \quad \int_Y |N(y,x)| q(y) d\nu(y) \leq B p(x), \ x \text{ in } X.$$

Then the formula

$$(Tf)(y) = \int_X N(y,x) f(x) d\mu(x),$$

defines a bounded operator T from $L^2(\mu)$ to $L^2(\nu)$ with $\|T\| \leq \sqrt{AB}$.

We will also need several lemmas. The first, and the final conclusion in the second, are due to J. E. Shapiro [24].

Lemma 3.2 [24] *Let φ be an analytic self-map of D with Clark measures μ_α, $|\alpha| = 1$. If f is continuous on ∂D, then*

$$\lim_{r \nearrow 1} \int_{\partial D} f \ \frac{1 - r^2}{|\alpha - r\varphi|^2} \ \frac{d\theta}{2\pi} = \int_{\partial D} f \, d\mu_\alpha^s,$$

for $|\alpha| = 1$.

Lemma 3.3 *If f is continuous on ∂D, then*

$$\lim_{r \nearrow 1} \int_{\partial D} f \ \frac{1 - |r\varphi|^2}{|\alpha - r\varphi|^2} \ \frac{d\theta}{2\pi} = \int_{\partial D} f \, d\mu_\alpha,$$

uniformly in α, $|\alpha| = 1$. Thus $\int_{\partial D} f \, d\mu_\alpha$ is a continuous function of α.

Proof. For $0 < r < 1$, the function $\frac{1-|r\varphi|^2}{|\alpha - r\varphi|^2}$ is bounded and harmonic on D, and thus is the Poisson integral of its boundary function. It follows from this, and the definition of μ_α, that the conclusion holds for $f = P_z$, the Poisson kernel at any z in D. Thus the conclusion holds when f is a finite linear combination of Poisson kernels. Such finite linear combinations are uniformly dense in the continuous functions on ∂D, and the lemma follows. □

The third lemma is a variant of exercise (7) in §26 of Halmos' treatise [11]; for the proof the interested reader can easily adapt the hint given there.

Lemma 3.4 *Let f and f_n, for $n = 1, 2, 3, \ldots$, be non-negative integrable functions on a measure space (X, μ) and suppose $\lambda \geq 0$. If $f_n \to f$, μ-a.e. as $n \to \infty$, and*

$$\limsup_{n\to\infty} \left\{ \int_X f_n d\mu - \int_X f\, d\mu \right\} \leq \lambda,$$

then

$$\limsup_{n\to\infty} \int_X |f_n - f| d\mu \leq \lambda.$$

Proof of Theorem 3.1. We can write $w = |w|v$ where v is measurable and unimodular on ∂D. Since $M_w = M_v M_{|w|}$ and M_v is unitary, the theorem is unaltered by assuming that $w = |w| \geq 0$. For now we also assume that w is continuous on ∂D.

Consider the normalized kernel function

$$K_a(z) = \frac{\sqrt{1 - |a|^2}}{1 - \overline{a}z}, \qquad |a| < 1.$$

If $|a| \to 1$, then $K_a \to 0$ weakly in H^2. Now take $a = r\alpha$ where $0 \leq r < 1$ and α is fixed in ∂D. Then $|C_\varphi K_{r\alpha}|^2 = \frac{1-r^2}{|\alpha - r\varphi|^2}$ and

$$\int |w|^2 d\mu_\alpha^s = \lim_{r\nearrow 1} \int |w|^2 \frac{1 - r^2}{|\alpha - r\varphi|^2} \, \frac{d\theta}{2\pi} = \limsup_{r\nearrow 1} \|M_w C_\varphi K_{r\alpha}\|^2$$

where Lemma 3.2 gives the first equality; the lower bound for $\|M_w C_\varphi\|_e^2$ follows from Eqn. (2.3). (Throughout the proof, all integrals are taken over ∂D.)

For the upper bound we consider $M_w C_\varphi$ as an integral operator from $L^2 = L^2\left(\partial D, \frac{d\theta}{2\pi}\right)$ to $L^2\left(G, \frac{d\theta}{2\pi}\right)$, where $G = \{e^{i\theta} : w(e^{i\theta}) > 0\}$. By Eqn. (3.1) the kernel of this operator is

$$K(e^{i\theta}, e^{it}) = w(e^{i\theta}) \, \frac{1 - |\varphi(e^{i\theta})|^2}{|e^{it} - \varphi(e^{i\theta})|^2}.$$

Similarly, if $0 < r < 1$, the kernel of the integral operator $M_w C_{r\varphi}$ is

$$K_r(e^{i\theta}, e^{it}) = w(e^{i\theta}) \, \frac{1 - |r\varphi(e^{i\theta})|^2}{|e^{it} - r\varphi(e^{i\theta})|^2}.$$

Since $\|r\varphi\|_\infty \leq r < 1$, $C_{r\varphi}$, and thus $M_w C_{r\varphi}$, are compact. We apply the Schur test to the integral operator $M_w C_{r\varphi} - M_w C_\varphi$, which has integral kernel $N = K_r - K$. We take $\mu = \frac{d\theta}{2\pi}$, ν to be the restriction of $\frac{d\theta}{2\pi}$ to G, $p(e^{i\theta}) = 1$ and $q(e^{i\theta}) = w(e^{i\theta})$. Then

$$\begin{aligned}
&\int |K_r(e^{i\theta}, e^{it}) - K(e^{i\theta}, e^{it})| p(e^{it}) \frac{dt}{2\pi} \\
&\leq \; w(e^{i\theta}) \int \left(\frac{1 - |r\varphi(e^{i\theta})|^2}{|e^{it} - r\varphi(e^{i\theta})|^2} + \frac{1 - |\varphi(e^{i\theta})|^2}{|e^{it} - \varphi(e^{i\theta})|^2} \right) \frac{dt}{2\pi} \\
&= \; 2w(e^{i\theta}) = 2q(e^{i\theta}),
\end{aligned}$$

for all $e^{i\theta}$ in the circle. This is the first Schur hypothesis.

For the second Schur hypothesis, we write

$$\lambda = \sup_{|\alpha|=1} \int |w|^2 d\mu_\alpha^s,$$

and consider sequences $r_n \nearrow 1$ and $\{\alpha_n\}$ in ∂D. It is enough to show that

$$\limsup_{n\to\infty} \int |K_{r_n}(e^{i\theta}, \alpha_n) - K(e^{i\theta}, \alpha_n)| w(e^{i\theta}) \frac{d\theta}{2\pi} \leq 2\lambda, \tag{3.3}$$

for then, since $M_w C_{r_n\varphi}$ is compact, the Schur test with $A = 2$ and $B = 2\lambda$ will imply that

$$\|M_w C_\varphi\|_e^2 \leq \limsup_{n\to\infty} \|M_w C_{r_n\varphi} - M_w C_\varphi\|^2 \leq 4\lambda,$$

as desired.

We may assume that α_n tends to some α in ∂D as $n \to \infty$. First we use Lemma 3.4 with

$$f_n = |w|^2 \frac{1 - |\varphi|^2}{|\alpha_n - \varphi|^2}, \ f = |w|^2 \frac{1 - |\varphi|^2}{|\alpha - \varphi|^2}.$$

Note that

$$\int f \frac{d\theta}{2\pi} = \int |w|^2 d\mu_\alpha - \int |w|^2 d\mu_\alpha^s,$$

and similarly for f_n, μ_{α_n} and $\mu_{\alpha_n}^s$, so that

$$\int (f_n - f) \frac{d\theta}{2\pi} = \int |w|^2 d\mu_{\alpha_n} - \int |w|^2 d\mu_\alpha + \int |w|^2 d\mu_\alpha^s - \int |w|^2 d\mu_{\alpha_n}^s.$$

By the final conclusion in Lemma 3.3, the difference of the first two terms tends to zero as $n \to \infty$, so that

$$\limsup_{n\to\infty} \int (f_n - f) \frac{d\theta}{2\pi} \leq \lambda.$$

Since $f_n \to f$ a.e., Lemma 3.4 implies that

$$\limsup_{n\to\infty} \int |K(e^{i\theta}, \alpha_n) - K(e^{i\theta}, \alpha)| w(e^{i\theta}) \frac{d\theta}{2\pi} \leq \lambda. \tag{3.4}$$

Now use Lemma 3.4 again, this time with

$$f_n = |w|^2 \ \frac{1 - |r_n\varphi|^2}{|\alpha_n - r_n\varphi|^2}, \ f = |w|^2 \ \frac{1 - |\varphi|^2}{|\alpha - \varphi|^2}.$$

We have

$$\begin{aligned}\int (f_n - f)\frac{d\theta}{2\pi} &\leq \left|\int |w|^2 \ \frac{1-|r_n\varphi|^2}{|\alpha_n - r_n\varphi|^2}\ \frac{d\theta}{2\pi} - \int |w|^2 d\mu_{\alpha_n}\right| \\ &+ \left|\int |w|^2 d\mu_{\alpha_n} - \int |w|^2 d\mu_{\alpha}\right| \\ &+ \left|\int |w|^2 d\mu_{\alpha} - \int |w|^2 \ \frac{1-|\varphi|^2}{|\alpha - \varphi|^2}\ \frac{d\theta}{2\pi}\right|.\end{aligned}$$

As $n \to \infty$, the first two terms on the right tend to zero by Lemma 3.3, while the last term is exactly $\int |w|^2 d\mu_\alpha^s$, which does not exceed λ. Thus, according to Lemma 3.4,

$$\limsup_{n\to\infty} \int |K_{r_n}(e^{i\theta},\alpha_n) - K(e^{i\theta},\alpha)| w(e^{i\theta})\frac{d\theta}{2\pi} \leq \lambda.$$

Combining this with Eqn. (3.4) yields Eqn. (3.3) and thus the desired upper bound for $\|M_w C_\varphi\|_e^2$.

It remains to consider the case where $w = |w|$ is continuous at each point of $E(\varphi)$, but not necessarily on all of ∂D. Since $E(\varphi)$ is compact, there exists a function g continuous on ∂D with $g = w$ on $E(\varphi)$. Then $w - g$ is continuous at and vanishes at every point of $E(\varphi)$. Given $\epsilon > 0$ one can construct a function h continuous on ∂D with $h = \epsilon$ on $E(\varphi)$ and $|w - g| \leq h$ on ∂D. Since $\|M_{w-g} C_\varphi f\| \leq \|M_h C_\varphi f\|$ for all f in L^2, $\|M_{w-g} C_\varphi\|_e \leq \|M_h C_\varphi\|_e$ by Eqn. (2.3). An application of this inequality and the triangle inequality yields

$$\|M_g C_\varphi\|_e - \|M_h C_\varphi\|_e \leq \|M_w C_\varphi\|_e \leq \|M_g C_\varphi\|_e + \|M_h C_\varphi\|_e.$$

Because g and h are both continuous on ∂D, we can apply our earlier argument to estimate $\|M_g C_\varphi\|_e$ and $\|M_h C_\varphi\|_e$; in particular, $\|M_h C_\varphi\|_e = O(\epsilon)$ as $\epsilon \to 0$. Since ϵ is arbitrary and $w = g$ on $E(\varphi)$, the theorem follows. □

4 Local Compact Difference and a Sum Theorem for H^2

Our plan for studying linear combinations in $\mathcal{B}(H^2)/\mathcal{K}$ is to decompose, mod $\mathcal{K}$, a composition operator into pieces associated to subsets of $E(\varphi)$ in a manner analogous to the decomposition in Theorem B above. This depends on an H^2 analogue of Theorem C above using Theorem 3.1 and H^2 versions of ideas from [16]. Throughout, φ and ψ are analytic self-maps of D with $|\varphi| < 1$ a.e. and $|\psi| < 1$ a.e. on ∂D, $\rho = \left|\frac{\varphi-\psi}{1-\overline{\varphi}\psi}\right|$, and $\varphi_t = t\varphi + (1-t)\psi$, $0 \leq t \leq 1$. We require three lemmas.

Lemma 4.1 *Let $a < 1$ and suppose G is a measurable subset of ∂D with $\rho \le a$ on G. Assume that $0 \le t \le 1$. Then $M_{\chi_G} M_{\varphi-\psi} C_{\varphi_t}$ acts boundedly from A_1^2 to L^2, and*

$$\|M_{\chi_G} M_{\varphi-\psi} C_{\varphi_t}\|_{\mathcal{B}(A_1^2, L^2)} \le \frac{c}{1-a} \|M_{\chi_G} M_\rho C_{\varphi_t}\|_{\mathcal{B}(H^2, L^2)},$$

where c is an absolute constant. Moreover, the same inequality (with a different c) holds if both norms are replaced by the corresponding essential norms. In particular, if the operator $M_{\chi_G} M_\rho C_{\varphi_t} : H^2 \to L^2$ is compact, so is $M_{\chi_G} M_{\varphi-\psi} C_{\varphi_t} : A_1^2 \to L^2$.

Proof. We consider the measures

$$\nu_t = \left(\chi_G |\varphi - \psi|^2 \frac{d\theta}{2\pi} \right) \circ \varphi_t^{-1}, \ \beta_t = \left(\chi_G \rho^2 \frac{d\theta}{2\pi} \right) \circ \varphi_t^{-1}$$

on the disk D. As in [16] we have

$$\begin{aligned} \frac{1 - |\varphi_t|^2}{|1 - \overline{\varphi}\psi|} &= \left| 1 + \overline{\varphi} \frac{(\psi - \varphi_t)}{1 - \overline{\varphi}\psi} + \varphi_t \frac{(\overline{\varphi} - \overline{\varphi}_t)}{1 - \overline{\varphi}\psi} \right| \\ &= \left| 1 - t\overline{\varphi} \frac{(\varphi - \psi)}{1 - \overline{\varphi}\psi} + (1-t)\varphi_t \frac{(\overline{\varphi} - \overline{\psi})}{1 - \overline{\varphi}\psi} \right| \\ &\ge 1 - \rho. \end{aligned}$$

Thus, if ζ lies in ∂D, $\delta > 0$ and $S(\zeta, \delta) = \{z \in D : |z - \zeta| < \delta\}$, on the set $G \cap \varphi_t^{-1}(S(\zeta, \delta))$ we have

$$|\varphi - \psi|^2 = \rho^2 |1 - \overline{\varphi}\psi|^2 \le \rho^2 \left(\frac{1 - |\varphi_t|^2}{1-a} \right)^2 \le \frac{4\delta^2}{(1-a)^2} \ \rho^2.$$

It follows that

$$\frac{\nu_t(S(\zeta,\delta))}{\delta^3} \le \frac{4}{(1-a)^2} \ \frac{\beta_t(S(\zeta,\delta))}{\delta},$$

so in the terminology of Section 2.5,

$$\Delta_3(\nu_t) \le \frac{4}{(1-a)^2} \ \Delta_1(\beta_t) \text{ and } \Delta_3^*(\nu_t) \le \frac{4}{(1-a)^2} \ \Delta_1^*(\beta_t).$$

Since for all bounded analytic functions h we have

$$\int_D |h|^2 d\nu_t = \|M_{\chi_G} M_{\varphi-\psi} C_{\varphi_t} h\|_{L^2}^2 \text{ and } \int_D |h|^2 d\beta_t = \|M_{\chi_G} M_\rho C_{\varphi_t} h\|_{L^2}^2,$$

the desired conclusion follows from Theorem 2.1 and the formula (2.3). $\square$

Lemma 4.2 *Let G be an open set in ∂D on which ρ is essentially bounded away from one, and assume that $E(\varphi) \cap E(\psi) \cap G$ is closed. Then $E(\varphi_t) \cap G$ is contained in $E(\varphi) \cap E(\psi) \cap G$ for $0 < t < 1$.*

Proof. For z in D we have

$$\frac{1 - |\varphi_t(z)|}{1 - |z|} \geq t \, \frac{1 - |\varphi(z)|}{1 - |z|} + (1 - t) \, \frac{1 - |\psi(z)|}{1 - |z|}.$$

On letting $z \to \zeta$ in ∂D, we see that if φ_t has a finite angular derivative at ζ, so do φ and ψ, that is, $F(\varphi_t) \subset F(\varphi) \cap F(\psi)$. The opposite containment holds by linearity of the angular derivative and so $F(\varphi_t) = F(\varphi) \cap F(\psi)$.

Suppose that I is an open arc whose closure lies in G and does not intersect both $E(\varphi)$ and $E(\psi)$. Let $\mu_{\alpha,t}$ be a Clark measure for φ_t. The point masses of $\mu_{\alpha,t}$, if any, are carried by $F(\varphi_t) \subset E(\varphi) \cap E(\psi)$, so that $\mu_{\alpha,t}$ puts no mass at the endpoints of I. A theorem of J. E. Shapiro [24] then states that

$$\mu^s_{\alpha,t}(I) = \lim_{r \nearrow 1} \int_I \frac{1 - r^2}{|\alpha - r\varphi_t|^2} \, \frac{d\theta}{2\pi}, \tag{4.1}$$

where $\mu^s_{\alpha,t}$ is the singular part of $\mu_{\alpha,t}$. It is enough to show that this quantity is zero, for then *spt* $\left(\mu^s_{\alpha,t}\right)$ cannot intersect I, and consequently neither can $E(\varphi_t)$.

For some $a < 1$ we have $\rho \leq a$ on G. Thus for $e^{i\theta}$ in G, $\psi(e^{i\theta})$ lies in the closed pseudo-hyperbolic disk with pseudo-hyperbolic radius a and pseudo-hyperbolic center $\varphi(e^{i\theta})$. On noting the Euclidean center and radius of this (also Euclidean) disk [8, p. 44], one can verify that there is a positive constant c, depending only on $a < 1$, such that

$$|\alpha - r\varphi_t(e^{i\theta})| \geq c \max\{|\alpha - r\varphi(e^{i\theta})|, |\alpha - r\psi(e^{i\theta})|\}$$

for all α in ∂D, $e^{i\theta}$ in G and $0 < r < 1$.

Given any ζ in the closure $\overline{I}$ of the above arc I, there exists an open arc $A(\zeta)$ in $\partial\mathbb{D}$ containing ζ with either $\overline{A(\zeta)} \cap E(\varphi)$ or $\overline{A(\zeta)} \cap E(\psi)$ empty. The open cover $\{A(\zeta) : \zeta \in \overline{I}\}$ for $\overline{I}$ has a finite subcover $\{A_1, \ldots, A_n, B_1, \ldots, B_m\}$ such that all of the sets $\overline{A}_i \cap E(\varphi)$, $i = 1, \ldots, n$ and $\overline{B}_j \cap E(\psi)$, $j = 1, \ldots, m$, are empty; of course, this subcover could consist only of A_i's, or only of B_j's. By Eqn. (4.1) and the previous paragraph,

$$\begin{aligned}
\mu^s_{\alpha,t}(I) &= \lim_{r \nearrow 1} \int_I \frac{1 - r^2}{|\alpha - r\varphi_t|^2} \, \frac{d\theta}{2\pi} \\
&\leq \lim_{r \nearrow 1} \frac{1}{c} \left[\sum_{i=1}^{n} \int_{A_i} \frac{1 - r^2}{|\alpha - r\varphi|^2} \, \frac{d\theta}{2\pi} + \sum_{j=1}^{m} \int_{B_j} \frac{1 - r^2}{|\alpha - r\psi|^2} \, \frac{d\theta}{2\pi} \right] \\
&= \frac{1}{c} \left[\sum_{i=1}^{n} \mu^s_{\alpha,0}(A_i) + \sum_{j=1}^{m} \mu^s_{\alpha,1}(B_j) \right] \\
&= 0,
\end{aligned}$$

as desired. $\square$

Our third lemma localizes Theorem 2.2. The proof is as in [17], but now uses the operator Equation (2.5) left-multiplied by M_{χ_G}.

Lemma 4.3 *Let G be a measurable subset of ∂D. Suppose that the weighted composition operators $M_{\chi_G} M_{\varphi-\psi} C_{\varphi_t}$ act boundedly from A_1^2 to L^2 with norms uniformly bounded for $0 \le t \le 1$. If $M_{\chi_G} M_{\varphi-\psi} C_{\varphi_t}$ is compact for $0 \le t \le 1$, then $M_{\chi_G}(C_\varphi - C_\psi)$ is compact from H^2 to L^2.*

We can now state our local compact difference theorem.

Theorem 4.4 *Let U be an open subset of ∂D whose boundary intersects neither $E(\varphi)$ nor $E(\psi)$. Suppose that ρ can be re-defined on a set of measure zero (if necessary), so that $\lim_{e^{i\theta} \to \zeta} \rho(e^{i\theta}) = \rho(\zeta) = 0$ for every ζ in $E(\varphi) \cap E(\psi) \cap U$. Then $M_{\chi_U}(C_\varphi - C_\psi)$ is a compact operator from H^2 to L^2.*

Proof. Since the intersection of $E(\varphi) \cap E(\psi)$ with U must be compact, there is an open subset G of U, containing this intersection and such that $\rho \le \frac{1}{2}$ on G. We have

$$M_{\chi_U}(C_\varphi - C_\psi) = M_{\chi_G}(C_\varphi - C_\psi) + M_{\chi_{U \setminus G}} C_\varphi - M_{\chi_{U \setminus G}} C_\psi.$$

The last two operators on the right are compact by Theorem 3.1. According to Lemma 4.3, compactness of $M_{\chi_G}(C_\varphi - C_\psi)$ will follow if we can show that the operators $M_{\chi_G} M_{\varphi-\psi} C_{\varphi_t}$ map A_1^2 into L^2, are uniformly bounded in norm and are each compact. Lemma 4.1 gives uniform boundedness since

$$\|C_{\varphi_t}\|^2_{\mathcal{B}(H^2)} \le \frac{2}{1 - |\varphi_t(0)|} \le 2 \max \left\{ \frac{1}{1 - |\varphi(0)|}, \frac{1}{1 - |\psi(0)|} \right\}, \tag{4.2}$$

see [8], while compactness follows from Lemmas 4.1, 4.2, and Theorem 3.1. $\square$

We close this section with a sum theorem for H^2. Here $\varphi, \varphi_1, \varphi_2, \ldots, \varphi_n$ are analytic self-maps of D with $|\varphi| < 1$ a.e. and $|\varphi_i| < 1$ a.e. on ∂D, $i = 1, 2, \ldots, n$.

Theorem 4.5 *Let $\varphi, \varphi_1, \ldots, \varphi_n$ be as above. Suppose that*

(a) The sets $E(\varphi_i)$, $i = 1, \ldots, n$ are pairwise disjoint, and $E(\varphi)$ coincides with $E(\varphi_1) \cup \cdots \cup E(\varphi_n)$, and

(b) The functions $\rho_i = \left| \frac{\varphi - \varphi_i}{1 - \overline{\varphi}\varphi_i} \right|$ can be altered on a set of measure zero in ∂D (if necessary) to guarantee that $\lim_{e^{i\theta} \to \zeta} \rho_i(e^{i\theta}) = \rho_i(\zeta) = 0$ for all ζ in $E(\varphi_i)$, $i = 1, \ldots, n$.

Then, as operators on H^2, $C_\varphi \equiv C_{\varphi_1} + \cdots + C_{\varphi_n}$ (mod $\mathcal{K}$).

Proof. Clearly, we can find pairwise disjoint open sets $U_1, \ldots, U_n$ in ∂D such that U_i contains $E(\varphi_i)$ and $\rho_i \leq \frac{1}{2}$ on U_i, $i = 1, \ldots, n$. Let $G = \partial D \setminus \bigcup_{i=1}^n U_i$, so that $\partial D = G \cup U_1 \cup \cdots \cup U_n$. For a measurable subset B of ∂D, let us write M_B for M_{χ_B}. Then

$$\begin{aligned}
&C_\varphi - C_{\varphi_1} - C_{\varphi_2} - \cdots - C_{\varphi_n} \\
&\quad= (M_G + M_{U_1} + \cdots + M_{U_n})(C_\varphi - C_{\varphi_1} - \cdots - C_{\varphi_n}) \\
&\quad= M_G C_\varphi - \sum_{i=1}^n M_G C_{\varphi_i} \\
&\quad+ M_{U_1}(C_\varphi - C_{\varphi_1}) - \sum_{i \neq 1} M_{U_1} C_{\varphi_i} \\
&\quad+ M_{U_2}(C_\varphi - C_{\varphi_2}) - \sum_{i \neq 2} M_{U_2} C_{\varphi_i} \\
&\quad+ \vdots \\
&\quad+ M_{U_n}(C_\varphi - C_{\varphi_n}) - \sum_{i \neq n} M_{U_n} C_{\varphi_i}.
\end{aligned}$$

Theorem 3.1 tells us that every individual term on the right is compact except possibly $M_{U_i}(C_\varphi - C_{\varphi_i}),\ i = 1, \ldots, n$. These, however, must be compact by Theorem 4.4. □

5 Linear Combinations mod $\mathcal{K}$

In this section we consider a class $\mathcal{S}$ of analytic self-maps of D for which definitive computations can be done. For $\varphi_1, \ldots, \varphi_n$ in $\mathcal{S}$, we determine which linear combinations $c_1 C_{\varphi_1} + \cdots + c_n C_{\varphi_n}$ are compact. We begin with some results which hold for arbitrary analytic self-maps of D.

5.1 A First-Order Lower Bound for $\|c_1 C_{\varphi_1} + \cdots + c_n C_{\varphi_n}\|_e$

Two analytic self-maps of D have the *same first-order data* at ζ in ∂D provided ζ lies in both $F(\varphi)$ and $F(\psi)$, $\varphi(\zeta) = \psi(\zeta)$ and $\varphi'(\zeta) = \psi'(\zeta)$. A special case of a theorem of MacCluer [13] states that if $C_{\varphi_1}, \ldots, C_{\varphi_n}$ act on $\mathcal{D}_\beta$, ζ is in ∂D, and no two of the maps $\varphi_1, \ldots, \varphi_n$ have the same first-order data at ζ, then

$$\|c_1 C_{\varphi_1} + \cdots + c_n C_{\varphi_n}\|_e^2 \geq \sum_{k=1}^n |c_k|^2 \frac{1}{|\varphi_k'(\zeta)|^\beta}.$$

Here it is understood that if ζ is not in $F(\varphi_k)$, then $|\varphi_k'(\zeta)| = \infty$. We will need a minor but useful extension of this result. For ζ in ∂D and $M > 0$, let $\Gamma^1_{\zeta,M}$ denote the curve in D given by $\frac{|z-\zeta|}{1-|z|^2} = M$, the boundary of a non-tangential approach region with vertex at ζ. At the point ζ the sides of this region make angle θ with the radius to ζ, where $2\cos\theta = \frac{1}{M}$. We will use the notation "$\lim_{\Gamma^1_{\zeta,M}}$" to indicate a limit taken as $z \to \zeta$ along the starboard leg of $\Gamma^1_{\zeta,M}$ (that is, *counterclockwise*).

Lemma 5.1 *Suppose φ and ψ are analytic self-maps of D. Then*

$$\lim_{\Gamma^1_{\zeta,M}} \frac{1-|z|^2}{1-\overline{\varphi(z)}\psi(z)}$$
$$= \begin{cases} \dfrac{2}{\overline{(1+i\tan\theta)}|\varphi'(\zeta)| + (1+i\tan\theta)|\psi'(\zeta)|} & \text{if } \zeta \in F(\varphi)\cap F(\psi) \\ & \text{and } \varphi(\zeta) = \psi(\zeta), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The Schwarz inequality applied to the H^2 kernel functions gives

$$\frac{1-|z|^2}{|1-\overline{\varphi(z)}\psi(z)|} \le \left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^{\frac{1}{2}} \left(\frac{1-|z|^2}{1-|\psi(z)|^2}\right)^{\frac{1}{2}}. \tag{5.1}$$

Both factors on the right are bounded in D, and if ζ is not in $F(\varphi)$, say, the first factor tends to zero as $z \to \zeta$. Thus we may assume that ζ lies in $F(\varphi)\cap F(\psi)$. It is also clear that if $\varphi(\zeta) \ne \psi(\zeta)$, the left side of (5.1) tends to zero as $z \to \zeta$ along $\Gamma^1_{\zeta,M}$.

In the remaining case, ζ lies in $F(\varphi)\cap F(\psi)$ and $\varphi(\zeta) = \psi(\zeta)$. For z in $\Gamma^1_{\zeta,M}$, we have

$$\frac{1-\overline{\varphi(z)}\psi(z)}{1-|z|^2} = \frac{1-|\varphi(z)|^2}{1-|z|^2} + \overline{\varphi(z)}\; M\; \frac{\varphi(z)-\psi(z)}{|\zeta - z|}.$$

Since $\Gamma^1_{\zeta,M}$ is nontangential at ζ, the first term on the right tends to $|\varphi'(\zeta)|$ as $z \to \zeta$ along $\Gamma^1_{\zeta,M}$. Moreover on the counterclockwise leg of $\Gamma^1_{\zeta,M}$, $\zeta - z \sim \zeta e^{i\theta}|\zeta - z|$ as $z \to \zeta$. Recall that $\zeta\overline{\varphi(\zeta)}\varphi'(\zeta) = |\varphi'(\zeta)|$ and similarly for ψ. Since $Me^{i\theta} = \frac{1}{2}(1+i\tan\theta)$ and $\varphi(\zeta) = \psi(\zeta)$, we have

$$\begin{aligned} \lim_{\Gamma^1_{\zeta,M}} \frac{1-\overline{\varphi(z)}\psi(z)}{1-|z|^2} &= |\varphi'(\zeta)| + M\zeta\overline{\varphi(\zeta)}e^{i\theta}(\psi'(\zeta)-\varphi'(\zeta)) \\ &= \frac{1}{2}\left[(1+i\tan\theta)|\psi'(\zeta)| + \overline{(1+i\tan\theta)}|\varphi'(\zeta)|\right], \end{aligned}$$

as desired. $\square$

For ζ in $F(\varphi)$ we call the vector $D_1(\varphi,\zeta) = (\varphi(\zeta), \varphi'(\zeta))$ the first-order data of φ at ζ. Suppose we fix analytic self-maps of D, $\varphi_1, \ldots, \varphi_n$. For ζ in ∂D, we denote by $\mathcal{D}_1(\zeta)$ the set of first-order data vectors at ζ associated to these self-maps:

$$\mathcal{D}_1(\zeta) = \{D_1(\varphi_j,\zeta) : 1 \le j \le n \text{ and } \zeta \in F(\varphi_j)\}.$$

Throughout Section 5 we write $F = F(\varphi_1) \cup \cdots \cup F(\varphi_n)$.

Theorem 5.2 *With the above notation, let $C_{\varphi_1}, \dots, C_{\varphi_n}$ act on $\mathcal{D}_\beta$. Then for any complex numbers $c_1, \dots, c_n$ and ζ in F,*

$$\|c_1 C_{\varphi_1} + \cdots c_n C_{\varphi_n}\|_e^2 \geq \sum_{\mathbf{d} \in \mathcal{D}_1(\zeta)} \Big| \sum_{\substack{\zeta \in F(\varphi_j) \\ D_1(\varphi_j, \zeta) = \mathbf{d}}} c_j \Big|^2 \frac{1}{|d_1|^\beta},$$

where $\mathbf{d} = (d_0, d_1)$.

Proof. Referring to Lemma 5.1, we see that if φ and ψ are analytic self-maps of D, then M tending to infinity means that $\theta \to \frac{\pi}{2}$, so that

$$\lim_{M \to \infty} \lim_{\Gamma^1_{\zeta, M}} \frac{1 - |z|^2}{1 - \overline{\varphi(z)}\psi(z)} = \begin{cases} \dfrac{1}{|\varphi'(\zeta)|} & \text{if } \zeta \in F(\varphi) \cap F(\psi) \text{ and} \\ & D_1(\varphi, \zeta) = D_1(\psi, \zeta), \\ 0 & \text{otherwise.} \end{cases}$$

As $|z| \to 1$ the normalized kernel functions $\frac{k_z}{\|k_z\|}$ tend weakly to zero in $\mathcal{D}_\beta$. Since $C_\varphi^* k_z = k_{\varphi(z)}$, we see from Eqn. (2.3) that

$$\begin{aligned}
&\|c_1 C_{\varphi_1} + \cdots + c_n C_{\varphi_n}\|_e^2 \\
&\quad \geq \lim_{M \to \infty} \lim_{\Gamma^1_{\zeta, M}} \left\| \left(\overline{c}_1 C^*_{\varphi_1} + \cdots + \overline{c}_n C^*_{\varphi_n} \right) \frac{k_z}{\|k_z\|} \right\|_e^2 \\
&\quad = \sum_{j, \ell = 1}^n \overline{c}_j c_\ell \lim_{M \to \infty} \lim_{\Gamma^1_{\zeta, M}} \left(\frac{1 - |z|^2}{1 - \overline{\varphi_j(z)}\varphi_\ell(z)} \right)^\beta \\
&\quad = \sum_{\substack{\zeta \in F(\varphi_j) \cap F(\varphi_\ell) \\ D_1(\varphi_j, \zeta) = D_1(\varphi_\ell, \zeta)}} c_j \overline{c}_\ell \frac{1}{|\varphi_j(\zeta)|^\beta},
\end{aligned}$$

which is a restatement of the desired conclusion. $\square$

Corollary 5.3 *If $c_1 C_{\varphi_1} + \cdots + c_n C_{\varphi_n}$ is compact on $\mathcal{D}_\beta$, then for every ζ in F and every $\mathbf{d}$ in $\mathcal{D}_1(\zeta)$,*

$$\sum_{\substack{\zeta \in F(\varphi_j) \\ D_1(\varphi_j, \zeta) = \mathbf{d}}} c_j = 0.$$

5.2 A Remark on Theorem B

Corollary 5.3 shows that the hypothesis on angular derivative sets in Theorem B is actually implied by condition (ii) of the theorem.

Corollary 5.4 *Suppose that $C_\varphi, C_{\varphi_1}, \ldots, C_{\varphi_n}$ act on $\mathcal{D}_\beta$ and*

$$C_\varphi \equiv C_{\varphi_1} + \cdots + C_{\varphi_n} \pmod{\mathcal{K}}.$$

Then $F(\varphi_1), \ldots, F(\varphi_n)$ are pairwise disjoint and $F(\varphi_1) \cup \cdots \cup F(\varphi_n)$ coincides with $F(\varphi)$.

Proof. Let us write $\varphi = \varphi_0$, $c_0 = 1$, and $c_j = -1$ for $j = 1, \ldots, n$, so that

$$c_0 C_{\varphi_0} + c_1 C_{\varphi_1} + \cdots + c_n C_{\varphi_n} \in \mathcal{K}.$$

By Corollary 5.3, each of the sets $\{j : \ \zeta \in F(\varphi_j) \text{ and } D_1(\varphi_j, \zeta) = \mathbf{d}\}$ is either empty or contains exactly two elements, namely zero and an integer from $\{1, \ldots, n\}$. The conclusion follows. $\square$

5.3 Lower Bounds from Higher-Order Data

We have considered the first-order data $D_1(\varphi, \zeta) = (\varphi(\zeta), \varphi'(\zeta))$ for ζ in $F(\varphi)$. In what follows, we look at higher-order data vectors

$$D_k(\varphi, \zeta) = (\varphi(\zeta), \varphi'(\zeta), \varphi''(\zeta), \ldots, \varphi^{(k)}(\zeta))$$

at points where the corresponding derivatives make sense. Specifically, we say φ has k^{th}-order data at ζ in ∂D if there exist complex numbers $b_0, b_1, \ldots, b_k$ with $|b_0| = 1$ so that

$$\varphi(z) = b_0 + b_1(z - \zeta) + \cdots + b_k(z - \zeta)^k + o(|z - \zeta|^k)$$

as $z \to \zeta$ unrestrictedly in D. In this case $\lim_{\angle z \to \zeta} \varphi^{(j)}(z)$ exists and equals $j! b_j$ for $j = 1, \ldots, n$ (see, for example, the argument on p. 47 in [23]); we refer to this limit as $\varphi^{(j)}(\zeta)$. Since $|b_0| = 1$, ζ is in $F(\varphi)$ and b_1 is the angular derivative $\varphi'(\zeta)$.

The model for this definition is of course a map which continues analytically across ∂D near ζ. Aside from the partial Taylor expansion we want our φ to inherit another property of analyticity: order of contact. We say an analytic self-map φ of D has *order of contact* $c > 0$ at ζ if $|\varphi(\zeta)| = 1$ and

$$\frac{1 - |\varphi(e^{i\theta})|^2}{|\varphi(\zeta) - \varphi(e^{i\theta})|^c}$$

is essentially bounded above and away from zero as $e^{i\theta} \to \zeta$. To clarify this and subsequent calculations, we map to the upper half-plane $\Omega = \{w : \ \mathrm{Im}\, w > 0\}$. For

any α in ∂D, consider the conformal map $\tau_\alpha(z) = i\,\frac{\alpha - z}{\alpha + z}$, which takes D onto Ω and α to 0. If φ has finite angular derivative at ζ, then $u = \tau_{\varphi(\zeta)} \circ \varphi \circ \tau_\zeta^{-1}$ is an analytic self-map of Ω having non-tangential limit $u(0) = 0$ and finite angular derivative $u'(0) = |\varphi'(\zeta)|$. Suppose for the moment that φ has an analytic continuation to a neighborhood of ζ, so the same is true of u at the origin. For w near zero, $u(w) = \sum_{n=1}^\infty a_n w^n$ with $a_1 = u'(0)$. If we assume $|\varphi| < 1$ a.e. on ∂D, then

$$\text{Im } u(x) = \sum_{n=1}^{\infty} (\text{Im } a_n) x^n$$

is positive for real x near (but not equal to) zero. The smallest natural number n with a_n non-real must be even, say $n = 2m$, and so Im $u(x) \sim (\text{Im } a_{2m})x^{2m}$ as $x \to 0$; moreover, Im $a_{2m} > 0$. If follows that near zero the image of $\mathbb{R}$ under u is approximated by the curve $y = cx^{2m}$ for appropriate $c > 0$. Further, if $\tau_\zeta(e^{i\theta}) = x$, then

$$\text{Im } u(x) = \frac{1 - |\varphi(e^{i\theta})|^2}{|\varphi(\zeta) + \varphi(e^{i\theta})|^2},$$

and we find that $\frac{1-|\varphi(e^{i\theta})|^2}{|\varphi(\zeta)-\varphi(e^{i\theta})|^{2m}}$ tends to a positive number as $e^{i\theta} \to \zeta$, so that φ has order of contact $2m$ at ζ.

Definition 5.5 We say an analytic self-map φ of D has *sufficient data* at ζ in ∂D if

(i) φ has finite angular derivative at ζ;

(ii) φ has order of contact $2m$ at ζ for some natural number m;

(iii) φ has $2m^{th}$-order data at ζ.

If φ has sufficient data at ζ with order of contact $2m$ and $u = \tau_{\varphi(\zeta)} \circ \varphi \circ \tau_\zeta^{-1}$ is as above, then u has an analogous expansion at the origin,

$$u(w) = \sum_{j=0}^{2m} \frac{u^{(j)}(0)}{j!}\; w^j + o(|w|^{2m});$$

here the derivatives can be realized as the non-tangential limits $\lim_{\angle w \to 0} u^{(j)}(w)$, $j = 0, 1, \ldots, 2m$. Moreover, if $1 \le k \le 2m$, the k^{th} order data vector $D_k(\varphi, \zeta)$ determines and is determined by the corresponding data $u'(0), \ldots, u^{(k)}(0)$ of u at zero.

Given ζ in ∂D, a natural number $k \ge 2$ and $M > 0$, let $\Gamma^k_{\zeta,M}$ denote the locus of the equation $\frac{|\zeta - z|^k}{1-|z|^2} = M$ in D, a curve having "order of contact k" with ∂D at ζ. We write "$\lim_{\Gamma^k_{\zeta,M}}$" to indicate a limit taken as z tends to ζ along $\Gamma^k_{\zeta,M}$.

Lemma 5.6 *Suppose analytic self-maps of D, φ and ψ, both have sufficient data at ζ in ∂D. Let $u = \tau_{\varphi(\zeta)} \circ \varphi \circ \tau_\zeta^{-1}$ and $v = \tau_{\psi(\zeta)} \circ \psi \circ \tau_\zeta^{-1}$, so that for w near zero in Ω,*

$$u(w) = \sum_{j=1}^{2m} \frac{u^{(j)}(0)}{j!}\, w^j + o(|w|^{2m}), \qquad v(w) = \sum_{j=1}^{2n} \frac{v^{(j)}(0)}{j!}\, w^j + o(|w|^{2n}),$$

where $2m$ and $2n$ are the respective orders of contact of φ and ψ at ζ. Then, if $k \geq 2$ and $M > 0$,

$$\lim_{\Gamma^k_{\zeta,M}} \frac{1-|z|^2}{1-\overline{\varphi(z)}\psi(z)} = \left[\overline{\left(\frac{u'(0)}{2} - i\frac{M}{2^{k-1}k!}u^{(k)}(0)\right)} + \left(\frac{v'(0)}{2} - i\frac{M}{2^{k-1}k!}v^{(k)}(0)\right)\right]^{-1}$$

provided $k \leq 2m$, $k \leq 2n$, and $D_{k-1}(\varphi,\zeta) = D_{k-1}(\psi,\zeta)$, while the limit is zero otherwise.

Proof. First we assume that $2m \geq k$, $2n \geq k$ and $D_{k-1}(\varphi,\zeta) = D_{k-1}(\psi,\zeta)$. We put $\alpha = \varphi(\zeta) = \psi(\zeta)$ and observe from direct calculation that with $w = \tau_\zeta(z)$,

$$\frac{v(w) - \overline{u(w)}}{2i \text{ Im } w} = \frac{1-\overline{\varphi(z)}\psi(z)}{1-|z|^2} \cdot \frac{|\zeta + z|^2}{(\overline{\alpha} + \overline{\varphi(z)})(\alpha + \psi(z))}.$$

If we let $z \to \zeta$ along the curve $\Gamma^k_{\zeta,M}$, $w = \tau_\zeta(z)$ tends to zero along its image in Ω, which is a slight enough perturbation of the curve $\tilde{\Gamma}_k$ defined by the equation $\frac{|w|^k}{\text{Im } w} = \frac{M}{2^{k-2}}$ that the latter can be used to compute our limit. That is,

$$\lim_{\Gamma^k_{\zeta,M}} \frac{1-\overline{\varphi(z)}\psi(z)}{1-|z|^2} = \lim_{\tilde{\Gamma}^k} \frac{v(w) - \overline{u(w)}}{2i \text{ Im } w}.$$

Moreover,

$$\frac{v(w) - \overline{u(w)}}{2i \text{ Im } w} = \frac{\text{Im } u(w)}{\text{Im } w} + \frac{v(w) - u(w)}{2i \text{ Im } w}. \tag{5.2}$$

We write

$$u(w) = \sum_{j=1}^{2m} a_j w^j + o(|w|^{2m}),\ v(w) = \sum_{j=1}^{2n} b_j w^j + o(|w|^{2n}),$$

and put $w = re^{i\theta}$. Consider the first term on the right in Eqn. (5.2). Since $a_1, a_2, \ldots, a_{2m-1}$ are real and $\text{Im } a_{2m} > 0$, we have

$$\begin{aligned} \text{Im } u(w) &= a_1 r \sin\theta + a_2 r^2 \sin 2\theta + \cdots + a_{2m-1} r^{2m-1} \sin(2m-1)\theta \\ &+ |a_{2m}| r^{2m} \sin(t_{2m} + 2m\theta) + o(r^{2m}), \end{aligned}$$

where $a_{2m} = |a_{2m}|e^{it_{2m}}$ with $0 < t_{2m} < \pi$. We take w in $\tilde{\Gamma}^k$ and divide by $\mathrm{Im}\, w = r\sin\theta = \frac{2^{k-2}}{M}\, r^k$ to see that

$$\lim_{\tilde{\Gamma}_k} \frac{\mathrm{Im}\ u(w)}{\mathrm{Im}\ w} = a_1 + (\mathrm{Im}\ a_k)\frac{M}{2^{k-2}}, \quad 2 \le k \le 2m, \tag{5.3}$$

where we use the fact that a_k is real if $k < 2m$.

Now consider the second term in Eqn. (5.2). Since $D_{k-1}(\varphi,\zeta) = D_{k-1}(\psi,\zeta)$, we have $a_j = b_j$ for $j < k$. Thus for w in $\tilde{\Gamma}_k$,

$$\frac{v(w) - u(w)}{2i\ \mathrm{Im}\ w} = -i\ \frac{M}{2^{k-1}}(b_k - a_k)e^{ik\theta} + o(1).$$

Since $\theta \to 0$ as $w \to 0$ along $\tilde{\Gamma}_k$,

$$\lim_{\tilde{\Gamma}_k} \frac{v(w) - u(w)}{2i\ \mathrm{Im}\ w} = -i\ \frac{M}{2^{k-1}}(b_k - a_k).$$

This equation, Eqn. (5.3) and Eqn. (5.2) give the desired result.

Suppose now that $2 \le k \le \min\{2m, 2n\}$ and $D_{k-1}(\varphi,\zeta) \ne D_{k-1}(\psi,\zeta)$. Let p be the smallest integer with $p \le k-1$ and $a_p \ne b_p$. Then for $w = re^{i\theta}$ in $\tilde{\Gamma}_k$,

$$\frac{v(w) - u(w)}{2i\ \mathrm{Im}\ w} = -i\ \frac{M}{2^{k-1}}\left(\frac{b_p - a_p}{r^{k-p}}\right)e^{ip\theta} + o\left(\frac{1}{r^{k-p}}\right),$$

a quantity whose modulus tends to infinity as $w \to 0$ along $\tilde{\Gamma}_k$. Thus

$$\lim_{\Gamma^k_{\zeta,M}} \frac{1 - |z|^2}{1 - \overline{\varphi(z)}\psi(z)} = 0. \tag{5.4}$$

Finally, if $k > 2m$, Eqn. (5.4) follows from Eqn. (5.1), the definition of "order of contact" and simple estimates applied to the Clark measure inequality

$$\frac{1 - |\varphi(z)|^2}{|\varphi(\zeta) - \varphi(z)|^2} \ge \int_{\partial D} P_z\ \frac{1 - |\varphi|^2}{|\varphi(\zeta) - \varphi|^2}\ \frac{d\theta}{2\pi}.$$

The case $k > 2n$ is similar. $\square$

Now fix analytic self-maps $\varphi_1, \ldots, \varphi_n$ of D and ζ in F. Assume that any φ_j having finite angular derivative at ζ (that is, ζ is in $F(\varphi_j)$) in fact has sufficient data at ζ. Given an integer $k \ge 2$, we write $\mathbb{M}_k(\zeta)$ for the set of those integers j, $1 \le j \le n$, for which $F(\varphi_j)$ contains ζ and the order of contact of φ_j at ζ is at least k. Let us write $\mathcal{D}_k(\zeta) = \{D_k(\varphi_j,\zeta) :\ j \in \mathbb{M}_k(\zeta)\}$. We have a higher-order analogue of Theorem 5.2.

Theorem 5.7 *Assume that $\varphi_1, \dots, \varphi_n$ are analytic self-maps of D as described above, and let ζ be in F. If $c_1, \dots, c_n$ are complex, $k \geq 3$ and notation is as above,*

$$\|c_1 C_{\varphi_1} + \cdots + c_n C_{\varphi_n}\|_e^2 \geq \sum_{\mathbf{d} \in \mathcal{D}_{k-1}(\zeta)} \Big| \sum_{\substack{j \in \mathbb{M}_k(\zeta) \\ D_{k-1}(\varphi_j, \zeta) = \mathbf{d}}} c_j \Big|^2 \frac{1}{|d_1|^\beta},$$

where $\mathbf{d} = (d_0, d_1, \dots, d_{k-1})$ and each C_{φ_j} acts on $\mathcal{D}_\beta$.

Proof. First consider analytic self-maps φ and ψ of D, each of which has sufficient data, with respective orders of contact $2m$ and $2n$, at a given ζ in ∂D. Let u and v be related to φ and ψ as in Lemma 5.6. Since $u^{(j)}(0)$ is real for $1 \leq j < 2m$ and Im $u^{(2m)}(0) > 0$ (and similarly for $v^{(j)}(0)$), we see from Lemma 5.6 that

$$\lim_{M \to \infty} \lim_{\Gamma^{k-1}_{\zeta, M}} \frac{1 - |z|^2}{1 - \overline{\varphi(z)}\psi(z)} = \begin{cases} \dfrac{1}{|\varphi'(\zeta)|} & \text{if } k \leq 2m,\ k \leq 2n, \text{ and} \\ & D_{k-1}(\varphi, \zeta) = D_{k-1}(\psi, \zeta), \\ 0 & \text{otherwise.} \end{cases}$$

Proceeding as in the proof of Theorem 5.2, we find

$$\begin{aligned} \|c_1 C_{\varphi_1} + \cdots + c_n C_{\varphi_n}\|_e^2 \ &\geq \ \sum_{j, \ell = 1}^{n} \overline{c_j} c_\ell \lim_{M \to \infty} \lim_{\Gamma^{k-1}_{\zeta, M}} \left(\frac{1 - |z|^2}{1 - \overline{\varphi_j(z)}\varphi_\ell(z)} \right)^\beta \\ &= \sum_{\substack{j, \ell \in \mathbb{M}_k(\zeta) \\ D_{k-1}(\varphi_j, \zeta) = D_{k-1}(\varphi_\ell, \zeta)}} \overline{c_j} c_\ell \, \frac{1}{|\varphi_j'(\zeta)|^\beta}, \end{aligned}$$

which is the desired conclusion. $\square$

The above theorem yields a higher-order version of MacCluer's lower bound for $\|C_\varphi - C_\psi\|_e$ in [13].

Corollary 5.8 *Fix ζ in ∂D and analytic self-maps φ and ψ of D, both of which have sufficient data at ζ, with respective orders of contact $2m$ and $2n$.*

(i) If $n < m$, then $\|C_\varphi - C_\psi\|_e^2 \geq \frac{1}{|\varphi'(\zeta)|^\beta}$.

(ii) If $n = m$ and $D_{2m-1}(\varphi, \zeta) \neq D_{2m-1}(\psi, \zeta)$, then

$$\|C_\varphi - C_\psi\|_e^2 \geq \frac{1}{|\varphi'(\zeta)|^\beta} + \frac{1}{|\psi'(\zeta)|^\beta}.$$

Proof. Apply Theorem 5.7 with $\varphi_1 = \varphi$, $\varphi_2 = \psi$, $c_1 = 1$, and $c_2 = -1$. $\square$

We need a more delicate version of Theorem 5.7, which is conveniently expressed in terms of the following formalism. For $\beta \geq 1$, $\mathcal{D}^+_\beta$ will denote the reproducing kernel Hilbert space of functions on the right half-plane $\Omega^+ = \{z : \text{ Re } z > 0\}$ having the kernel functions $k^+_w(z) = (z + \overline{w})^{-\beta}$, w in Ω^+. These spaces appear in the literature with various defining normalizations. When $\beta = 1$, $\mathcal{D}^+_\beta$ is the Hardy space on Ω^+, see [12]. For $\beta > 1$, $\mathcal{D}^+_\beta$ is the weighted Bergman space of all functions f analytic on Ω^+ for which

$$\|f\|^2_{\mathcal{D}^+_\beta} \equiv \frac{2^{\beta-2}(\beta-1)}{\pi} \int_{\Omega^+} |f(x+iy)|^2 x^{\beta-2} dx\, dy$$

is finite, see [21, p. 74]. For us a key fact (used in Section 5.5) is this: $\{k^+_w : w \in \Omega_+\}$ *is a linearly independent set in* $\mathcal{D}^+_\beta$.

Now let $\mathbb{M}_k(\zeta)$ and $\mathcal{D}_k(\zeta)$ be as defined prior to Theorem 5.7.

Lemma 5.9 *Let $\varphi_1, \ldots, \varphi_n$ be analytic self-maps of D. Fix ζ in F and suppose that if $F(\varphi_j)$ contains ζ, then φ_i has sufficient data at ζ, $j = 1, \ldots, n$. Let u_j be related to φ_j as u is related to φ in Lemma 5.6. Then if $A > 0$ and k is an even natural number,*

$$\|c_1 C_{\varphi_1} + \cdots + c_n C_{\varphi_n}\|^2_e \;\; \geq \sum_{\mathbf{d} \in \mathcal{D}_{k-1}(\zeta)} \Bigg\| \sum_{\substack{j \in \mathbb{M}_k(\zeta) \\ D_{k-1}(\varphi_j, \zeta) = \mathbf{d}}} \overline{c_j} k^+_{\frac{u'_j(0)}{2} - iAu_j^{(k)}(0)} \Bigg\|^2_{\mathcal{D}^+_\beta}.$$

Proof. We use Lemma 5.6 with $M = A2^{k-1}k!$. Then we have

$$\begin{aligned}
&\|c_1 C_{\varphi_1} + \cdots + c_n C_{\varphi_n}\|^2_e \\
&\geq \lim_{\Gamma^k_{\zeta,M}} \left\| (\overline{c_1} C^*_{\varphi_1} + \cdots + \overline{c_n} C^*_{\varphi_n}) \frac{k_z}{\|k_z\|} \right\|^2_{\mathcal{D}_\beta} \\
&\geq \lim_{\Gamma^k_{\zeta,M}} \sum \overline{c_j} c_\ell \left(\frac{1 - |z|^2}{1 - \overline{\varphi_j(z)} \varphi_\ell(z)} \right)^\beta \\
&= \sum_{\substack{j,\ell \in \mathbb{M}_k(\zeta) \\ D_{k-1}(\varphi_j,\zeta) = D_{k-1}(\varphi_\ell,\zeta)}} \overline{c_j} c_\ell \left[\overline{\left(\frac{u'_j(0)}{2} - iAu_j^{(k)}(0) \right)} + \left(\frac{u'_\ell(0)}{2} - iAu_\ell^{(k)}(0) \right) \right]^{-\beta} \\
&= \sum_{\mathbf{d} \in \mathcal{D}_{k-1}(\zeta)} \Bigg\| \sum_{\substack{j \in \mathbb{M}_k(\zeta) \\ D_{k-1}(\varphi_j,\zeta) = \mathbf{d}}} \overline{c_j} k^+_{\frac{u'_j(0)}{2} - iAu_j^{(k)}(0)} \Bigg\|^2_{\mathcal{D}^+_\beta},
\end{aligned}$$

as desired. Note that since $u_j'(0) > 0$, $A > 0$ and Im $u_j^{(k)}(0) \geq 0$, the complex number $\frac{u_j(0)}{2} - iAu_j^{(k)}(0)$ lies in Ω^+. $\square$

5.4 The Class $\mathcal{S}$ and Making $\|C_\varphi - C_\psi\|_e$ Small

For maps φ and ψ with sufficient data at a given ζ in ∂D, Corollary 5.8 describes two obstructions to $\|C_\varphi - C_\psi\|_e$ being small:

(a) unequal orders of contact at ζ;

(b) equal order of contact $2m$ but $D_{2m-1}(\varphi, \zeta) \neq D_{2m-1}(\psi, \zeta)$.

In this section we estimate $\|C_\varphi - C_\psi\|_e$ in the absence of these obstructions and characterize when it is zero. We work within the class $\mathcal{S}$ of analytic self-maps φ of D for which $E(\varphi)$ is a finite set (so that $E(\varphi) = F(\varphi)$) and such that φ has sufficient data at each point of $F(\varphi)$. For simplicity we restrict attention to composition operators on $\mathcal{D}_1 = H^2$. We write Ω for the upper half-plane $\{z : \text{ Im } z > 0\}$; the pseudo-hyperbolic metric Λ on Ω is given by

$$\Lambda(z, w) = \left| \frac{z - w}{z - \overline{w}} \right|.$$

Note that $0 \leq \Lambda < 1$ on $\Omega \times \Omega$. Recall from Section 5.3 that if φ has sufficient data at ζ in $F(\varphi)$ with order of contact $2m$, and $u = \tau_{\varphi(\zeta)} \circ \varphi \circ \tau_\zeta^{-1}$, then $u^{(2m)}(0)$ lies in Ω.

Proposition 5.10 *Fix ζ in ∂D and suppose that φ and ψ have sufficient data with respective orders of contact $2m$ and $2n$ at ζ, and moreover that $\varphi(\zeta) = \psi(\zeta)$. Write $\rho = \left|\frac{\varphi - \psi}{1 - \overline{\varphi}\psi}\right|$, $u = \tau_{\varphi(\zeta)} \circ \varphi \circ \tau_\zeta^{-1}$ and $v = \tau_{\psi(\zeta)} \circ \psi \circ \tau_\zeta^{-1}$.*

(a) If $2m \neq 2n$, or $2m = 2n$ and $D_{2m-1}(\varphi, \zeta) \neq D_{2m-1}(\psi, \zeta)$, then $\rho(e^{i\theta}) \to 1$ as $e^{i\theta} \to \zeta$.

(b) If $2m = 2n$ and $D_{2m-1}(\psi, \zeta) = D_{2m-1}(\varphi, \zeta)$, then

$$\lim_{e^{i\theta} \to \zeta} \rho(e^{i\theta}) = \Lambda\left(u^{(2m)}(0), v^{(2m)}(0)\right) < 1.$$

(c) $\rho(e^{i\theta}) \to 0$ as $e^{i\theta} \to \zeta$ if and only if $2m = 2n$ and $D_{2m}(\varphi, \zeta) = D_{2m}(\psi, \zeta)$.

Proof. First assume that the hypotheses of (b) hold. Since $\psi(\zeta) = \varphi(\zeta)$, direct computation shows that if $\tau_\zeta(e^{i\theta}) = x$,

$$\rho(e^{i\theta}) = \left| \frac{u(x) - v(x)}{u(x) - \overline{v(x)}} \right|.$$

For $i = 1, 2, \ldots, 2m-1$, the MacLaurin coefficient a_i for u is real and equal to the corresponding coefficient b_i for v. On the other hand, $a_{2m} = u^{(2m)}(0)/(2m)!$ and $b_{2m} = v^{(2m)}(0)/(2m)!$ lie in Ω. Clearly,

$$\lim_{x\to 0}\left|\frac{u(x)-v(x)}{u(x)-\overline{v(x)}}\right| = \Lambda(a_{2m}, b_{2m}),$$

which is the desired conclusion for (b).

Clearly (b) implies (c); the interested reader can easily verify (a). $\square$

For φ, ψ, and ζ as in Proposition 5.10, let us use the convention that $\rho(\zeta) = \lim_{e^{i\theta}\to\zeta}\rho(e^{i\theta})$. The limit exists by Proposition 5.10 and for φ, ψ in $\mathcal{S}$ this convention, at worst, redefines ρ on a finite set, leaving the multiplication operator M_ρ unaltered. Note, however, that $\rho(\zeta)$ as just defined is in general not the same as the non-tangential limit $\lim_{\angle z\to\zeta}\rho(z)$.

Theorem 5.11 *Suppose that φ and ψ are in $\mathcal{S}$ with $F(\varphi) = F(\psi) = F$, and that C_φ and C_ψ act on H^2. Suppose that at each ζ in F, φ and ψ have common order of contact $2m(\zeta)$ with $D_{2m(\zeta)-1}(\varphi,\zeta) = D_{2m(\zeta)-1}(\psi,\zeta)$ and moreover that $\rho(\zeta) \le \frac{1}{2}$. Then*

$$\frac{1}{4}\max_{\zeta\in F}\frac{1}{|\varphi'(\zeta)|}\rho(\zeta)^2 \le \|C_\varphi - C_\psi\|_e^2 \le B\max_{\alpha\in\varphi(F)}\sum_{\varphi(\zeta)=\alpha}\frac{1}{|\varphi'(\zeta)|}\rho(\zeta)^2,$$

where B is an absolute constant and $\varphi(F) = \{\varphi(\zeta) : \zeta \in F\}$. Moreover, the lower bound is valid without the assumption that $\rho(\zeta) \le \frac{1}{2}$, $\zeta \in F$.

Note: Suppose that φ and ψ satisfy the hypotheses of the theorem, and that μ_α and ν_α are their respective Clark measures. Then μ_α^s and ν_α^s are pure point, and since φ and ψ have the same first-order data,

$$\nu_\alpha^s = \mu_\alpha^s = \sum_{\substack{\zeta\in F\\ \varphi(\zeta)=\alpha}}\frac{1}{|\varphi'(\zeta)|}\delta_\zeta.$$

The upper bound for $\|C_\varphi - C_\psi\|_e^2$ in the theorem then takes the form

$$B\sup_{|\alpha|=1}\int\rho^2 d\mu_\alpha^s.$$

We expect that there is a lower bound of the same type.

Proof of Theorem 5.11. For ζ in F write $u_\zeta = \tau_{\varphi(\zeta)}\circ\varphi\circ\tau_\zeta^{-1}$ and $v_\zeta = \tau_{\psi(\zeta)}\circ\psi\circ\tau_\zeta^{-1}$. According to Lemma 5.9, if ζ is in F and $A > 0$,

$$\|C_\varphi - C_\psi\|_e^2 \ge \left\|k^+_{\frac{u'_\zeta(0)}{2} - iAu_\zeta^{(2m(\zeta))}(0)} - k^+_{\frac{v'_\zeta(0)}{2} - iAv_\zeta^{(2m(\zeta))}(0)}\right\|^2_{H^2_+}.$$

A calculation analogous to [8, Lemma 9.11] shows that for z, w in Ω_+,

$$\|k_z^+ - k_w^+\|_{H_+^2}^2 = \left|\frac{z-w}{z+\overline{w}}\right|^2 \left(\frac{1}{z+\overline{z}} + \frac{1}{w+\overline{w}}\right).$$

We put

$$A = \frac{u_\zeta'(0)}{\left|u_\zeta^{(2m(\zeta))}(0) - \overline{v_\zeta^{(2m(\zeta))}(0)}\right|},$$

$$z = \frac{u_\zeta'(0)}{2} - iAu_\zeta^{(2m(\zeta))}(0),\ w = \frac{v_\zeta'(0)}{2} - iAv_\zeta^{(2m(\zeta))}(0),$$

and note that $|z+\overline{w}| \le 2u_\zeta'(0)$ while $|z-w| = u_\zeta'(0)\Lambda\left(u_\zeta^{(2m(\zeta))}(0), v_\zeta^{(2m(\zeta))}(0)\right)$. It follows that

$$\|k_z^+ - k_w^+\|_{H_+^2}^2 \ge \frac{1}{4u_\zeta'(0)}\,\Lambda\left(u_\zeta^{(2m(\zeta))}(0), v_\zeta^{(2m(\zeta))}(0)\right)^2,$$

which, by Proposition 5.10 and since $u_\zeta'(0) = |\varphi'(\zeta)|$ gives the lower bound for the essential norm of $C_\varphi - C_\psi$.

For the upper bound select a finite union G of pairwise disjoint open arcs which contains F and is such that $\rho \le \frac{2}{3}$ on G. Now

$$C_\varphi - C_\psi = M_{\chi_G}(C_\varphi - C_\psi) + M_{\chi_{\partial D\setminus G}}C_\varphi - M_{\chi_{\partial D\setminus G}}C_\psi,$$

and the last two terms on the right are compact operators by Theorem 3.1. Therefore $\|C_\varphi - C_\psi\|_e = \|M_{\chi_G}(C_\varphi - C_\psi)\|_e$ which we now estimate. To use Eqn. (2.3) it is permissible to restrict the sequence $\{f_n\}$ to a subspace of finite codimension. With notation as in Section 2.6, we pick a sequence of unit vectors f_n in $H_0^2 = (\ker X)^\perp$ which converges weakly to zero. When applied to an H^2 function, the operator identity (2.5) holds pointwise in D and thus, since the kernel functions span H^2, in the weak operator topology. Thus

$$\|M_{\chi_G}(C_\varphi - C_\psi)f_n\|_{L^2} \le \|X\|_{\mathcal{B}(H_0^2,A_1^2)} \int_0^1 \|M_{\chi_G}M_{\varphi-\psi}C_{\varphi_t}g_n\|_{L^2}dt,$$

where $g_n = Xf_n/\|Xf_n\|$. Since X is bounded below on H_0^2, $g_n \to 0$ weakly in A_1^2. Now $\|M_{\chi_G}M_{\varphi-\psi}C_{\varphi_t}\|_{\mathcal{B}(A_1^2,L^2)}$ is bounded for $0 \le t \le 1$ by Lemma 4.1 and Eqn. (4.2), so we can apply Fatou's Lemma to conclude that

$$\limsup_{n\to\infty}\|M_{\chi_G}(C_\varphi - C_\psi)f_n\|_{L^2} \le \|X\|_{B(H_0^2,A_1^2)} \int_0^1 \limsup_{n\to\infty}\|M_{\chi_G}M_{\varphi-\psi}C_{\varphi_t}g_n\|_{L^2}dt.$$

Moreover, by Eqn. (2.3) and Lemma 4.1 we have

$$\begin{aligned}\limsup_{n\to\infty}\|M_{\chi_G}M_{\varphi-\psi}C_{\varphi_t}g_n\|_{L^2} &\le \|M_{\chi_G}M_{\varphi-\psi}C_{\varphi_t}\|_{e,\mathcal{B}(A_1^2,L^2)} \\ &\le b\|M_{\chi_G}M_\rho C_{\varphi_t}\|_{e,\mathcal{B}(H^2,L^2)} \\ &\le b\|M_hC_{\varphi_t}\|_{e,\mathcal{B}(H^2,L^2)},\end{aligned}$$

where $b > 0$, h is any continuous function on ∂D with $h \geq \chi_G \rho$ on ∂D and $h(\zeta) = \rho(\zeta)$ for ζ in F. Clearly we can choose such an h while also requiring that $\rho \leq \frac{3}{4}$ on the set $W = \{e^{i\theta} : h(e^{i\theta}) > 0\}$. Then, by Theorem 3.1 and Lemma 3.2, if $\mu_{\alpha,t}$ is a Clark measure for φ_t,

$$\|M_h C_{\varphi_t}\|_e \leq 2 \sup_{|\alpha|=1} \left\{ \int_W h^2 d\mu_{\alpha,t}^s \right\}^{\frac{1}{2}} = 2 \sup_{|\alpha|=1} \left\{ \lim_{r \nearrow 1} \int_W h^2 \frac{1-r^2}{|\alpha - r\varphi_t|^2} \frac{d\theta}{2\pi} \right\}^{\frac{1}{2}}.$$

Since $\rho \leq \frac{3}{4}$ on W we see (as noted in the proof of Lemma 4.2) that there is a constant $c > 0$ such that $|\alpha - r\varphi_t(e^{i\theta})| \geq c|\alpha - r\varphi(e^{i\theta})|$, for α in ∂D, $e^{i\theta}$ in W, $0 < r < 1$ and $0 \leq t < 1$. Thus

$$\begin{aligned} \|M_h C_{\varphi_t}\|_e &\leq \frac{2}{c} \sup_{|\alpha|=1} \left\{ \lim_{r \nearrow 1} \int_{\partial D} h^2 \frac{1-r^2}{|\alpha - r\varphi|^2} \frac{d\theta}{2\pi} \right\}^{\frac{1}{2}} \\ &= \frac{2}{c} \sup_{|\alpha|=1} \int_{\partial D} h^2 d\mu_\alpha^s, \end{aligned}$$

where $\{\mu_\alpha\}$ are the Clark measures for φ. Since each μ_α^s is pure point in the present circumstances,

$$\mu_\alpha^s = \begin{cases} \displaystyle\sum_{\substack{\zeta \in F \\ \varphi(\zeta)=\alpha}} \frac{1}{|\varphi'(\zeta)|} \delta_\zeta & \text{if } \alpha \in \varphi(F), \\ 0 & \text{otherwise.} \end{cases}$$

In view of Eqn. (2.3) and the arbitrariness of $\{f_n\}$ in H_0^2, this gives the desired upper bound for $\|C_\varphi - C_\psi\|_e$. $\square$

Corollary 5.12 *Suppose that φ and ψ lie in $\mathcal{S}$ with respective orders of contact $2m(\zeta)$ and $2n(\zeta)$ at each ζ in (respectively) $F(\varphi)$ and $F(\psi)$. Then the following are equivalent.*

(i) $C_\varphi - C_\psi$ is compact on H^2.

(ii) $F(\psi) = F(\varphi)$ (we call this set F) and for all ζ in F, $2n(\zeta) = 2m(\zeta)$ and $D_{2m(\zeta)}(\psi, \zeta) = D_{2m(\zeta)}(\varphi, \zeta)$.

5.5 Linear Relations Mod $\mathcal{K}$

We fix $\varphi_1, \ldots, \varphi_n$ in $\mathcal{S}$ and again write F for the union $F(\varphi_1) \cup \cdots \cup F(\varphi_n)$, a finite set. For ζ in F and $k = 2, 4, 6, \ldots$, let

$$\mathbb{N}_k(\zeta) = \{j : F(\varphi_j) \text{ contains } \zeta \text{ and } k \text{ is the order of contact of } \varphi_j \text{ at } \zeta\}.$$

We also write $\mathcal{E}_k(\zeta) = \{D_k(\varphi_j, \zeta) : j \text{ is in } \mathbb{N}_k(\zeta)\}$.

Theorem 5.13 *Let $\varphi_1, \ldots, \varphi_n$ be in $\mathcal{S}$ and set notation as above. Given complex numbers $c_1, \ldots, c_n$, the following are equivalent:*

(i) $c_1 C_{\varphi_1} + \cdots + c_n C_{\varphi_n}$ is compact on $\mathcal{D}_\beta$;

(ii) for every ζ in F, every even $k \geq 2$ and every $\mathbf{d}$ in $\mathcal{E}_k(\zeta)$,

$$\sum_{\substack{j \in \mathbb{N}_k(\zeta) \\ D_k(\varphi_j, \zeta) = \mathbf{d}}} c_j = 0.$$

Proof. First assume that $c_1 C_{\varphi_1} + \cdots + c_n C_{\varphi_n}$ is compact. Fix ζ in F and let u_j be related to φ_j as u is related to φ in Lemma 5.6. Let $\mathbb{M}_k(\zeta)$ and $\mathcal{D}_k(\zeta)$ be as defined prior to Theorem 5.7. According to Lemma 5.9 (with $A = 1$), if $k \geq 2$, $\mathbf{d}$ is in $\mathcal{D}_{k-1}(\zeta)$, and $w_j = \frac{u_j'(0)}{2} - i u_j^{(k)}(0)$, then

$$\sum_{\substack{j \in \mathbb{M}_k(\zeta) \\ D_{k-1}(\varphi_j, \zeta) = \mathbf{d}}} \overline{c_j} k_{w_j}^+ = 0.$$

But if $D_{k-1}(\varphi_j, \zeta)$ and $D_{k-1}(\varphi_\ell, \zeta)$ coincide, $w_j = w_\ell$ exactly when $D_k(\varphi_j, \zeta) = D_k(\varphi_\ell, \zeta)$. Using the fact that $\{k_w^+ : \text{ Re } w > 0\}$ is linearly independent in $\mathcal{D}_\beta^+$, we see that for every $\mathbf{d}$ in $\mathcal{D}_k(\zeta)$,

$$\sum_{\substack{j \in \mathbb{M}_k(\zeta) \\ D_k(\varphi_j, \zeta) = \mathbf{d}}} c_j = 0. \tag{5.5}$$

Since $\mathbb{M}_k(\zeta)$ is the union of the disjoint sets $\mathbb{N}_k(\zeta)$ and $\mathbb{M}_{k+1}(\zeta)$, we see from Eqn. (5.5) that

$$\sum_{\substack{j \in \mathbb{N}_k(\zeta) \\ D_k(\varphi_j, \zeta) = \mathbf{d}}} c_j + \sum_{\substack{j \in \mathbb{M}_{k+1}(\zeta) \\ D_k(\varphi_j, \zeta) = \mathbf{d}}} c_j = 0.$$

Suppose now $\mathbf{d} = (d_0, d_1, \ldots, d_k)$. There are a finite number of elements in $\mathcal{D}_{k+1}(\zeta)$ having the form $(d_0, d_1, \ldots, d_k, *)$, call them $\mathbf{d}_i = (d_0, d_1, \ldots, d_k, d_{k+1}^i)$, $i = 1, 2, \ldots, r$. Then

$$\sum_{\substack{j \in \mathbb{M}_{k+1}(\zeta) \\ D_k(\varphi_j, \zeta) = \mathbf{d}}} c_j = \sum_{i=1}^r \Big\{ \sum_{\substack{j \in \mathbb{M}_{k+1}(\zeta) \\ D_{k+1}(\varphi_j, \zeta) = \mathbf{d}_i}} c_j \Big\}$$

and each of the r summands on the right vanishes, again by Eqn. (5.5). Therefore (ii) holds.

Now assume that (ii) holds. We consider the case $\beta = 1$. The set F is finite, say $F = \{\zeta_1, \ldots, \zeta_s\}$. Let $U_1, \ldots, U_s$ be disjoint arcs whose union is ∂D and for which the interior of U_i contains ζ_i for $i = 1, \ldots, s$. Clearly,

$$\sum_{j=1}^{n} c_j C_{\varphi_j} = \sum_{i=1}^{s} M_{\chi_{U_i}} \left(\sum_{j=1}^{n} c_j C_{\varphi_j} \right).$$

Taking U to be any U_i and $\zeta = \zeta_i$, it is enough to show that the operator

$$M_{\chi_U} \left(\sum_{j=1}^{n} c_j C_{\varphi_j} \right),$$

considered as acting from H^2 to L^2, is compact. This operator can be written as

$$\sum_{\zeta \notin F(\varphi_j)} c_j M_{\chi_U} C_{\varphi_j} + \sum_{\zeta \in F(\varphi_j)} c_j M_{\chi_U} C_{\varphi_j}. \tag{5.6}$$

If $F(\varphi_j)$ does not contain ζ, then χ_U is identically zero in a neighborhood of $F(\varphi_j)$, and we see from Theorem 3.1 that $M_{\chi_U} C_{\varphi_j}$ is compact. Thus the first sum in the expression (5.6) is a compact operator.

Consider now the second sum, which can be rewritten as

$$\sum_{m \geq 1} \left\{ \sum_{j \in \mathbb{N}_{2m}(\zeta)} c_j M_{\chi_U} C_{\varphi_j} \right\};$$

the sum over m is of course finite since $\mathbb{N}_{2m}(\zeta)$ is empty for m large enough. For nonempty $\mathbb{N}_{2m}(\zeta)$,

$$\sum_{j \in \mathbb{N}_{2m}(\zeta)} c_j M_{\chi_U} C_{\varphi_j} = \sum_{\mathbf{d} \in \mathcal{E}_{2m}(\zeta)} \Bigg\{ \sum_{\substack{j \in \mathbb{N}_{2m}(\zeta) \\ D_{2m}(\varphi_j, \zeta) = \mathbf{d}}} c_j M_{\chi_U} C_{\varphi_j} \Bigg\}.$$

If $\mathbf{d}$ is in $\mathcal{E}_{2m}(\zeta)$, j and ℓ are in $\mathbb{N}_{2m}(\zeta)$ and $D_{2m}(\varphi_j, \zeta) = D_{2m}(\varphi_\ell, \zeta) = \mathbf{d}$, Proposition 5.10(c) tells us that

$$\left| \frac{\varphi_j(e^{i\theta}) - \varphi_\ell(e^{i\theta})}{1 - \overline{\varphi_j(e^{i\theta})} \varphi_\ell(e^{i\theta})} \right| \to 0 \text{ as } e^{i\theta} \to \zeta.$$

It follows from Theorem 4.4 that $M_{\chi_U}(C_{\varphi_j} - C_{\varphi_\ell})$ is compact. Fix such ℓ and write $T = M_{\chi_U} C_{\varphi_\ell}$. Then for each j in $\mathbb{N}_{2m}(\zeta)$ with $D_{2m}(\varphi_j, \zeta) = \mathbf{d}$, there is a compact operator K_j with $M_{\chi_U} C_{\varphi_j} = T + K_j$. Thus

$$\sum_{\substack{j \in \mathbb{N}_{2m}(\zeta) \\ D_{2m}(\varphi_j, \zeta) = \mathbf{d}}} c_j M_{\chi_U} C_{\varphi_j} = \Bigg(\sum_{\substack{j \in \mathbb{N}_{2m}(\zeta) \\ D_{2m}(\varphi_j, \zeta) = \mathbf{d}}} c_j \Bigg) T + \sum_{\substack{j \in \mathbb{N}_{2m}(\zeta) \\ D_{2m}(\varphi_j, \zeta) = \mathbf{d}}} c_j K_j.$$

The coefficient of T vanishes by the hypothesis (ii), and we are left with a compact operator, verifying (i). The proof for the case $\beta > 1$ is similar, with Theorem C playing the role of Theorem 4.4, the sets U_i taken to lie in D, and Proposition 5.10(c) replaced by the assertion that if $2m = 2n$ and $D_{2m}(\varphi, \zeta) = D_{2m}(\psi, \zeta)$, then $\rho(z) \to 0$ as $z \to \zeta$ unrestrictedly in D, an implication easily established by calculations in the proofs of Lemma 5.6 and Proposition 5.10. $\square$

Remark 5.14 It is sometimes convenient to rephrase condition (ii) in Theorem 5.13. With notation as in the statement, fix ζ in F, an even natural number k and a vector $\mathbf{d}$ in $\mathcal{E}_k(\zeta)$. We define the vector $\mathbf{x}(\zeta, k, \mathbf{d}) = (a_1, \ldots, a_n)$ in $\mathbb{C}^n$, where $a_j = 1$ if j is in $\mathbb{N}_k(\zeta)$ and $D_k(\varphi_j, \zeta) = \mathbf{d}$, while $a_j = 0$ otherwise. Let $\mathcal{M} = \mathcal{M}(\varphi_1, \ldots, \varphi_n)$ denote the linear span in $\mathbb{C}^n$ of all such vectors $\mathbf{x}(\zeta, k, \mathbf{d})$. Clearly, condition (ii) in Theorem 5.13 is equivalent to

$$\text{(ii)}' \qquad (c_1, \ldots, c_n) \in \mathcal{M}^{\perp}.$$

Put another way, if we define a linear transformation $\mathbb{A}: \mathbb{C}^n \to \mathcal{B}(\mathcal{D}_\beta)/\mathcal{K}$ by

$$\mathbb{A}(c_1, \ldots, c_n) = [c_1 C_{\varphi_1} + \cdots + c_n C_{\varphi_n}],$$

where $[B]$ denotes the coset of the operator B, then the content of Theorem 5.13 is that $\ker \mathbb{A} = \mathcal{M}^{\perp}$. This yields the immediate:

Corollary 5.15 *Fix $\varphi_1, \ldots, \varphi_n$ in $\mathcal{S}$. Then the vector subspace of $\mathcal{B}(\mathcal{D}_\beta)/\mathcal{K}$ spanned by the cosets $[C_{\varphi_1}], \ldots, [C_{\varphi_n}]$ has the same dimension as the subspace $\mathcal{M}(\varphi_1, \ldots, \varphi_n)$ of $\mathbb{C}^n$. In particular, these cosets are linearly independent if and only if $\mathcal{M}(\varphi_1, \ldots, \varphi_n) = \mathbb{C}^n$.*

For maps φ in $\mathcal{S}$ with order of contact uniformly two, the linear fractional self-maps of D play a special role. Let us write $\mathcal{S}(2)$ for the collection of those φ in $\mathcal{S}$ which have order of contact two at each point of $F(\varphi)$. Further, we denote by $\mathcal{L}$ the collection of linear fractional self-maps φ of D which are not automorphisms but which have $\|\varphi\|_\infty = 1$. We note that any linear fractional map ψ is determined by its second-order data $D_2(\psi, z_0) = (\psi(z_0), \psi'(z_0), \psi''(z_0))$ at any point z_0 of analyticity. Suppose now that φ is in $\mathcal{S}(2)$ and ζ_0 is a point in $F(\varphi)$. Let φ_0 be the unique linear fractional map with $D_2(\varphi_0, \zeta_0) = D_2(\varphi, \zeta_0)$. Since the curvature of the parametric curve $\varphi(e^{i\theta})$ is determined by second-order data, the circle $\varphi_0(\partial D)$ is necessarily the osculating circle of this curve at the point $\varphi(\zeta_0)$. Thus φ_0 maps D to D and lies in $\mathcal{L}$. The following result was established by the second author on the weighted Bergman spaces [16]; here we extend it to H^2.

Corollary 5.16 *Let φ be in $\mathcal{S}(2)$ with $F(\varphi) = \{\zeta_1, \ldots, \zeta_r\}$. For $i = 1, \ldots, r$ let φ_i be the unique linear fractional map with $D_2(\varphi_i, \zeta_i) = D_2(\varphi, \zeta_i)$. Then*

$$C_\varphi \equiv C_{\varphi_1} + \cdots + C_{\varphi_r} \pmod{\mathcal{K}},$$

where the operators act on H^2.

Proof. This is immediate from Theorem 5.13 applied to the linear combination

$$C_\varphi - C_{\varphi_1} - C_{\varphi_2} - \cdots - C_{\varphi_r}.$$

□

Let us write $\mathcal{V}_2$ for the vector subspace of $\mathcal{B}(\mathcal{D}_\beta)/\mathcal{K}$ spanned by the cosets $[C_\varphi]$ with φ in $\mathcal{S}(2)$. According to Corollary 5.16, $\{[C_\varphi] : \varphi \in \mathcal{L}\}$ is a spanning subset of $\mathcal{V}_2$. That it is linearly independent as well follows immediately from Theorem 5.13, which gives the following:

Corollary 5.17 *$\{[C_\varphi] : \varphi \in \mathcal{L}\}$ is a basis for $\mathcal{V}_2$.*

In Theorem 5.13, the proof that (ii) implies (i) uses the local compact difference Theorem 4.4 to locally "match up" certain pairs C_{φ_j} and C_{φ_ℓ} by local data. As one changes the locality or local data, the pairs can change. However, the method suggests that with some minimality hypothesis, the coefficients in a linear equation

$$c_1 C_{\varphi_1} + \cdots + c_n C_{\varphi_n} \equiv 0 \pmod{\mathcal{K}}$$

should be integers up to a common scalar factor. The next result makes this precise.

Corollary 5.18 *Suppose that $\varphi, \varphi_1, \ldots, \varphi_n$ lie in $\mathcal{S}$ and assume that $[C_{\varphi_1}]$, …, $[C_{\varphi_n}]$ are linearly independent in $\mathcal{B}(\mathcal{D}_\beta)/\mathcal{K}$. If*

$$C_\varphi \equiv c_1 C_{\varphi_1} + \cdots + c_n C_{\varphi_n} \pmod{\mathcal{K}},$$

then $c_1, \ldots, c_n$ are rational numbers.

Proof. Let us write $\psi_1 = \varphi$ and $\psi_k = \varphi_{k-1}$, $b_1 = -1$ and $b_k = c_{k-1}$ for $k = 2, 3, \ldots, n+1$, so that the equation mod $\mathcal{K}$ in the statement becomes

$$b_1 C_{\psi_1} + b_2 C_{\psi_2} + \cdots + b_{n+1} C_{\psi_{n+1}} \equiv 0 \pmod{\mathcal{K}}.$$

Linear independence of $[C_{\psi_2}], \ldots, [C_{\psi_{n+1}}]$ implies that $b_1 = -1$ uniquely determines $b_2, \ldots, b_{n+1}$. Thus according to Remark 5.14, $\mathcal{M} = \mathcal{M}(\psi_1, \ldots, \psi_n)$ has codimension one in $\mathbb{C}^{n+1}$. From the vectors $\mathbf{x}(\zeta, k, \mathbf{d})$ which span $\mathcal{M}$, select a basis $\mathbf{x}_1, \ldots, \mathbf{x}_n$ for $\mathcal{M}$; the coordinates of each $\mathbf{x}_i$ are zeros and ones. On applying the Gram-Schmidt process to $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$, we obtain an orthonormal basis for $\mathcal{M}$ of the form $\mathbf{z}_i/\|\mathbf{z}_i\|$, $i = 1, 2, \ldots, n$, where each $\mathbf{z}_i$ has rational coefficients. Let P denote the orthogonal projection of $\mathbb{C}^{n+1}$ onto $\mathcal{M}$ and write $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{n+1}$ for the standard basis vectors in $\mathbb{C}^{n+1}$. For some j, $(I - P)\mathbf{e}_j$ is non-zero, and clearly

$$(I - P)\mathbf{e}_j = \mathbf{e}_j - \sum_{i=1}^{n} \frac{\langle \mathbf{e}_j, \mathbf{z}_i \rangle}{\|\mathbf{z}_i\|^2}\, \mathbf{z}_i$$

is a vector $(r_1, r_2, \ldots, r_{n+1})$ with obviously rational coefficients. Since $\mathcal{M}^\perp$ is one-dimensional, $(b_1, \ldots, b_{n+1}) = c(r_1, \ldots, r_{n+1})$ for some complex number c. Since $cr_1 = b_1 = -1$, c is rational so that $b_2, \ldots, b_{n+1}$ are rational as well. $\square$

Finally, we note that certain weighted composition operators enjoy a decomposition analogous to that in Corollary 5.16, with the sum replaced by a linear combination.

Proposition 5.19 *Let φ be in $\mathcal{S}(2)$ with $F(\varphi) = \{\zeta_1, \ldots, \zeta_r\}$ and suppose that φ_i, $i = 1, \ldots, r$, are linear fractional maps related to φ as in Corollary 5.16. If w is a bounded measurable function on ∂D which is continuous at each point of $F(\varphi)$, then*

$$M_w C_\varphi \equiv w(\zeta_1)C_{\varphi_1} + \cdots + w(\zeta_r)C_{\varphi_r} \ (mod\ \mathcal{K}),$$

where the operators are considered as mapping H^2 to L^2.

Proof. Choose pairwise disjoint arcs $I_1, \ldots, I_r$ with union $\partial\mathbb{D}$ and the interior of I_j containing ζ_j for $j = 1, \ldots, r$. We write χ_j for χ_{I_j}. We have

$$M_w C_\varphi = \sum_{j=1}^{r} \left[M_{(w - w(\zeta_j))\chi_j} C_\varphi + w(\zeta_j) M_{\chi_j} C_\varphi \right].$$

The first term in each summand on the right is compact by Theorem 3.1. Moreover the operators

$$M_{\chi_j}(C_\varphi - C_{\varphi_j}) \text{ and } M_{(1-\chi_j)} C_{\varphi_j}$$

are compact, the first by Theorem 4.4 and Proposition 5.10(c) and the second by Theorem 3.1; the proposition follows. $\square$

Note that left-multiplying all operators in the proposition by the orthogonal projection P of L^2 onto H^2 gives an alternate statement in which the multiplication operator M_w is replaced by the corresponding Toeplitz operator $T_w = PM_w|H^2$, and all operators act from H^2 to H^2.

6 Arc-Connectedness in $\mathcal{C}(\mathcal{S}_0)$

In [26], Shapiro and Sundberg posed the following interesting question: What is the relationship between the conditions

(a) $C_\varphi - C_\psi$ is compact;

(b) C_φ and C_ψ lie in the same component of the topological space of composition operators on $\mathcal{D}_\beta$?

The second author and Toews [17] proposed the scheme discussed in Section 2.6 and used it to give examples of composition operators satisfying (b) but not (a). Bourdon [3] presented analogous examples for linear fractional maps, showing that for φ and ψ in the class $\mathcal{L}$ (see Section 5.5), (a) corresponds to equal second-order data at the (common) point of contact with ∂D whereas equal first-order data characterizes (b). The second author [16] proved the corresponding result on the Bergman spaces for φ and ψ in the class $\mathcal{S}_0 \cap \mathcal{S}(2)$ where $\mathcal{S}_0$ is as defined below. Here we extend this result to H^2 and to higher orders of contact. First we record a general sufficient condition for (b), proved in [16] for the Bergman spaces.

Proposition 6.1 *Let us write $\varphi_t = t\varphi + (1-t)\psi$, $0 \le t \le 1$, where φ and ψ are analytic self-maps of D. If $\rho = \left|\frac{\varphi-\psi}{1-\overline{\varphi}\psi}\right|$ is essentially bounded away from one on ∂D, then there is a constant B such that*

$$\|C_{\varphi_s} - C_{\varphi_r}\|_{\mathcal{B}(H^2)} \le B|s-r|, \quad 0 \le r < s \le 1.$$

The proposition follows immediately from Theorem 2.2 and Lemma 4.1 with G taken to be the appropriate set of full measure.

We write $\mathcal{S}_0$ for the collection of those φ in $\mathcal{S}$ such that $\|\chi_{\partial D\setminus U}\varphi\|_\infty < 1$ for every open subset U of ∂D containing $F(\varphi)$. Such a φ is allowed to make contact with ∂D only at points of $F(\varphi)$. We also write $\mathcal{C}(\mathcal{S}_0)$ for the set of those C_φ with φ in $\mathcal{S}_0$.

Theorem 6.2 *Suppose that φ and ψ lie in $\mathcal{S}_0$ with respective orders of contact $2m(\zeta)$ and $2n(\zeta)$ at each ζ in (respectively) $F(\varphi)$ and $F(\psi)$. Let $\varphi_t = t\varphi+(1-t)\psi$, $0 \le t \le 1$. Then if $\beta \ge 1$, the following are equivalent:*

(i) C_φ and C_ψ lie in the same component of the space $\mathcal{C}(\mathcal{S}_0)$ equipped with the norm topology of $\mathcal{B}(\mathcal{D}_\beta)$.

(ii) There is a positive constant B such that

$$\|C_{\varphi_s} - C_{\varphi_r}\| \le B|s-r|, \quad 0 \le r < s \le 1.$$

(iii) $F(\varphi) = F(\psi)$ (call this set F), and for each ζ in F, $2m(\zeta) = 2n(\zeta)$ and $D_{2m(\zeta)-1}(\varphi,\zeta) = D_{2m(\zeta)-1}(\psi,\zeta)$.

Proof. First consider the H^2 case. Suppose that (iii) holds and write $\rho = \left|\frac{\varphi-\psi}{1-\overline{\varphi}\psi}\right|$ as usual. If ζ is in F, it is clear from Proposition 5.10 that $\lim_{e^{i\theta}\to\zeta}\rho(e^{i\theta}) < 1$. Thus there is an open set U in ∂D containing F such that $\|\chi_U\rho\|_\infty < 1$. Since φ and ψ lie in $\mathcal{S}_0$, $\|\chi_{\partial D\setminus U}\rho\|_\infty < 1$ as well, and (ii) follows from Proposition 6.1. For the Bergman space version of this implication, replace U by an appropriate subset of D and use the Bergman space analogue of Proposition 6.1 in [16], together with the following replacement for Proposition 5.10(b), a fact easily established by calculations in the

proofs of Lemma 5.6 and Proposition 5.10: If φ and ψ have sufficient data and common order of contact $2m$ at ζ in ∂D, and if $D_{2m-1}(\varphi, \zeta) = D_{2m-1}(\psi, \zeta)$, then

$$\limsup_{z \to \zeta} \rho(z) < 1,$$

the limit superior being taken unrestrictedly in D. Clearly (ii) implies (i) on any $\mathcal{D}_\beta$. Finally, we can use the topological argument of Theorem 2.4 in [13], with Corollary 5.8 playing the role of Theorem 2.2 in [13], to show that (i) implies (iii). □

The last theorem and Corollary 5.12 show that the phenomenon (a) ⇒ (b) (and not conversely) persists more broadly. Note, however, that by passing from the class $\mathcal{S}$ to $\mathcal{S}_0$, we have eliminated maps φ which touch the unit circle outside of $E(\varphi)$. Such points of contact are immaterial for condition (a); we do not know whether they matter for (b).

References

[1] A. Aleksandrov, Multiplicity of boundary values of inner functions, *Izv. Akad. Nauk Armyan. SSR Ser. Mat.* 22 (1987), no. 5, 490–503, 515.

[2] E. Berkson, Composition operators isolated in the uniform operator topology, *Proc. Amer. Math. Soc.* 81 (1981), 230–232.

[3] P. S. Bourdon, Components of linear-fractional composition operators, *J. Math. Anal. Appl.* 279 (2003), 228–245.

[4] P. S. Bourdon, D. Levi, S. K. Narayan, and J. H. Shapiro, Which linear fractional composition operators are essentially normal? *J. Math. Anal. Appl.* 280 (2003), 30–53.

[5] J. A. Cima and A. L. Matheson, Essential norms of composition operators and Aleksandrov measures, *Pacific J. Math.* 179 (1997), 59–64.

[6] D. N. Clark, One-dimensional perturbations of restricted shifts, *J. D'Analyse Math.* 25 (1972), 169–191.

[7] C. C. Cowen, Composition operators on H^2, *J. Operator Theory* 9 (1983), 77–106.

[8] C. C. Cowen and B. D. MacCluer, *Composition Operators on Spaces of Analytic Functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1995.

[9] P. L. Duren, *Theory of H^p Spaces*, Academic Press, New York, 1970.

[10] P. Gorkin and R. Mortini, Norms and essential norms of linear combinations of endomorphisms, *Trans. Amer. Math. Soc.* (to appear).

[11] P. R. Halmos, *Measure Theory*, Van Nostrand, New York, 1950.

[12] K. Hoffman, *Banach Spaces of Analytic Functions*, Prentice Hall, Englewood Cliffs, NJ, 1962.

[13] B. D. MacCluer, Components in the space of composition operators, *Integral Equations and Operator Theory* 12 (1989), 725–738.

[14] B. D. MacCluer, S. Ohno, and R. Zhao, Topological structure of the space of composition operators on H^∞, *Integral Equations and Operator Theory* 40 (2001), 481–494.

[15] B. D. MacCluer and J. H. Shapiro, Angular derivatives and compact composition operators on the Hardy and Bergman spaces, *Canadian J. Math.* 38 (1986), 878–906.

[16] J. Moorhouse, Compact differences of composition operators, *J. Functional Analysis* 219 (2005), 70–92.

[17] J. Moorhouse and C. Toews, Differences of composition operators, Trends in Banach Spaces and Operator Theory (Memphis, TN, 2001), 207–213, *Contemp. Math.* 321, Amer. Math. Soc., Providence, RI, 2003.

[18] N. K. Nikolski, *Operators, Functions, and Systems: An Easy Reading, Volume 1: Hardy, Hankel and Toeplitz*, Mathematical Surveys and Monographs, 92, Amer. Math. Soc., Providence, RI, 2002.

[19] P. J. Nieminen and E. Saksman, On the compactness of the difference of composition operators, *J. Math. Anal. Appl.* 298 (2004), 501–522.

[20] A. G. Poltoratski, Boundary behavior of pseudocontinuable functions (Russian), *Algebra i Analiz* 5 (1993), 189–210; translation in *St. Petersburg Math. J.* 5 (1994), 389–406.

[21] S. Saitoh, *Integral Transforms, Reproducing Kernels and their Applications*, Pitman Research Notes in Mathematics, 369, Longman, Harlow, 1997.

[22] D. E. Sarason, Composition operators as integral operators, *Analysis and Partial Differential Equations*, C. Sadosky (ed.), 545–565, Marcel-Dekker, New York, 1990.

[23] D. E. Sarason, *Sub-Hardy Hilbert Spaces in the Unit Disk*, University of Arkansas Lecture Notes in the Mathematical Sciences, Volume 10, John Wiley and Sons, New York, 1994.

[24] J. E. Shapiro, Aleksandrov measures used in essential norm inequalities for composition operators, *J. Operator Theory* 40 (1998), 133–146.

[25] J. H. Shapiro, The essential norm of a composition operator, *Annals Math.* 125 (1987), 375–404.

[26] J. H. Shapiro and C. Sundberg, Isolation amongst the composition operators, *Pacific J. Math.* 145 (1990), 117–152.

[27] J. H. Shapiro and C. Sundberg, Compact composition operators on L^1, *Proc. Amer. Math. Soc.* 108 (1990), 443–449.

[28] J. H. Shapiro and P. D. Taylor, Compact, nuclear and Hilbert-Schmidt composition operators on H^2, *Indiana Univ. Math. J.* 23 (1973), 471–496.